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Sensitivity Analysis of Semidefinite Programming Problems

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Zusammenfassung

Das Gebiet der semidefiniten Programmierung (SDP) ist eines der am aktivsten erforschten Gebiete der mathematischen Optimierung der letzten 20 Jahre. Bei der Sensitivitätsanalyse dieser Probleme, die zum Beispiel für das Konvergenzverhalten von Lösungsmethoden grosse Bedeutung hat, bleibt jedoch noch vieles unerforscht. Im Unterschied zu nicht-polyedrischen Kegeln, wie zum Beispiel die Kegel der semidefiniten Matrizen, ist die Sensitivitätsanalyse von Optimierungsproblemen über polyedrischen Kegeln gut abgedeckt. Um diese Lücke für SDP-Probleme zu schliessen, erweitern wir das Kalkül auf das Gebiet der nicht-polyedrischen Kegel.

In der vorliegenden Arbeit untersuchen wir parametrische Modelle von SDP-Problemen und analysieren das Störungsverhalten der Parameter. Verschiedene Konzepte der Lipschitz-Stabilität von stationären und zulässigen Punkten von SDP-Problemen werden diskutiert, namentlich die Calmness, die strenge Regularität und die lokal Lipschitz-Oberhalbstetigkeit. Zur Charakterisierung der Sensitivität arbeiten wir mit zwei unterschiedlichen Methoden der Variationsanalysis:
Der erste Ansatz verfolgt das Ziel, ein SDP-Problem als verallgemeinerte Abbildungen umzuschreiben. Davon werden verallgemeinerte Ableitungen gebildet, die an einem gegebenen Punkt injektiv sind, falls die gesuchte starke Stabilität oder lokal Lipschitz-Oberhalbstetigkeit gegeben ist. Wir konstruieren eine lokal Lipschitz-stetige Funktion der Karush-Kuhn-Tucker (KKT) Bedingungen, eine Kojima-ähnliche Funktion, des SDP-Problems. Die Kojima Funktion ist das Produkt einer stetig differenzierbaren und einer nichtglatten Funktion, wovon letztere die Projektion auf den Kegel der positiv semidefiniten Matrizen beinhaltet. Wir untersuchen die Konstruktion der Kontingent-Ableitung (graphische Ableitung) und der Thibault-Ableitung (strenge graphische Ableitung), und vergleichen die Thibault Ableitung mit der verallgemeinerten Jacobimatrix von Clarke.
Im zweiten Ansatz wird gezeigt, dass die Calmness-Eigenschaft direkt mit der Konvergenzgeschwindigkeit von einem Lösungs-Algorithmus zusammen

hängt. Wir betrachten ein SDP-Problem als ein endliches Ungleichungssystem mit rechts-seitigen Störungen und einer fixierten algebraischen Nebenbedingung. Für dieses gestörte System untersuchen wir die Calmness-Eigenschaft der Menge der stationären Lösungen und konstruieren einen algorithmischen Ansatz, mit dessen Hilfe die Calmness gezeigt werden kann.

In dieser Arbeit konstruieren und diskutieren wir die Thibault Ableitung für verschiedene Abbildungen in SDP-Problemen, leiten neue Ergebnisse zu Calmness der Projektionsabbildung her und geben hinreichende und notwendige Bedingungen für die Calmness-Eigenschaft von SDP-Problemen. Unsere Ergebnisse umfassen nicht nur SDP da sie teilweise sogar für allgemeinere nicht-polyedrische Kegel und konvexe Mengen anwendbar sind.

Abstract

Semidefinite programming (SDP) problems are among the most active areas of research in mathematical optimization in the past 20 years. However, sensitivity and stability analysis of the solution sets of SDP problems, which tells us about the behavior of solution methods, is still being uncovered.
While stability has been well covered for optimization problems with polyhedral cone constraints, several results for the nonpolyhedral case, such as the cones of semidefinite matrices, are still missing.
We try to close this gap in the SDP toolbox by extending the calculus to the case of nonpolyhedral cones.

In this thesis, we construct a parametrized model for the SDP problem and analyze the disturbance of the parameters. We discuss conditions for Lipschitz properties, namely: calmness, upper regularity, and strong regularity of nonlinear SDP problems at feasible and stationary points, and present results that even apply to more general nonpolyhedral cones and convex sets. We work with two different techniques from the field of variational analysis to show different sensitivity characteristics.
Our first approach is to rewrite the problem as a generalized function and construct generalized derivatives that are injective at a given solution if the stability of concern is given - upper and strong regularity. We construct a locally Lipschitz function of the Karush-Kuhn-Tucker conditions, a Kojima-like function, for an SDP problem. The Kojima function is the product of a continuously differentiable and a nonsmooth function. The latter contains the projection function onto the cone of positive semidefinite matrices. We look at the construction of its contingent derivative (graphical derivative) and Thibault derivative (strict graphical derivative). Moreover, we examine the relations between the Thibault derivative and the Clarke generalized Jacobian of these projections.
In the second approach, we show that calmness is directly related to the convergence speed of a solution algorithm. We look at an SDP problem as a mapping of a system of finitely many inequalities with perturbations on the

right-hand side and a fixed algebraic constraint. For this perturbed system, we study the calmness property of stationary points and create the structure of an algorithm to show calmness.

In this thesis, we construct and discuss the elements of the Thibault derivative for some mappings in SDP, give new results on calmness of the projection mapping, and present sufficient and necessary conditions for calmness in SDP problems. Our results even apply to more general problems, beyond SDP.

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1 Introduction

Semidefinite programming problems have arguably been the most active area of research on optimization in the past 20 years. While it can be seen as an extension of linear and nonlinear optimization problems, its history is almost as old as mathematical programming itself, and the tools and analysis of semidefinite matrices date back even further.
For the sake of brevity the abbreviation SDP will be used for *semidefinite programming* throughout this thesis.

1.1 History of SDP and Related Work

The study of the geometry of semidefinite cones goes back to the 1940s to Bohnenblust, eigenvalue problems even date back to Lagrange in 1773 (for overviews cf. Wolkowicz, Saigal, and Vandenberghe [53], and Lewis and Overton [35]).
Its recent popularity, since the 1990s, is both due to big advances in solving these problems, strong numerical results, and many diverse applications. Independent of each other, Nemirovskii and Nesterov, Alizadeh, and Karmarkar and Thakur developed efficient solution methods for SDP by extending polynomial-time interior-point methods for linear optimization to solve SDP problems (cf. [53] and references therein).
In the late 1980s, Nemirovskii and Nesterov implemented the first interior-point method for SDP. They extended many of the interior-point methods and theoretical results from linear programming to a much broader class of convex programming problems. They created polynomial-time algorithms for a wide range of convex optimization problems, and applied this theory successfully to semidefinite programming.
Alizadeh also proposed such generalizations for SDP. It appears that he introduced the terminology *semidefinite programming*, which has previously also been called *linear matrix inequalities* (LMI).
The success of these interior-point methods for SDP attracted researchers of interior-point methods in linear programming. The seminal compendium

Handbook of semidefinite programming: Theory, Algorithms, and Applications [53] gives an excellent overview of the research done in this field by the year 2000.
The burst of activities is partly due to the many applications such as control theory, combinatorial optimization, complexity, generalized convexity and nonlinear programming.
SDP solution methods give solutions of linear matrix inequalities in control theory, strong approximation results to hard combinatorial problems such as the max-cut problem.
Since linear SDP belongs to cone programming and convex optimization problems, other related areas are such diverse fields as e.g. traditional convex constrained optimization, statistics (covariance matrices, clustering), engineering, eigenvalue functions, financial applications etc. Because a linear SDP problem is a convex program it is solvable via interior-point methods to any desired accuracy in polynomial time. Most of these applications can usually be solved very efficiently in practice as well as in theory.
Interest has grown further in the beginning of the 21st century. There is a close connection between semidefinite matrices and polynomial optimization problems, hence, the field has expanded to algebraic geometry (cf. the handbook edited by Anjos and Lasserre [1]).

An important field in optimization is sensitivity and stability analysis. Over 100 years ago Lyapunov studied stability analysis of differential equations. For solutions to optimization problems over polyhedral cones this analysis is well covered (cf. Rockafellar and Wets [45], Bonnans and Shapiro [5], Klatte and Kummer [26], Faccchinei and Pang [14], Dontchev and Rockafellar [13]). However, when extended to nonpolyhedral cones such as in SDP problems a complete analysis of stability is still emerging.
Strong regularity conditions have been studied (cf. [8, 37, 42]) and different generalized derivatives have been constructed and analyzed (cf. [36, 48]).
In several recent papers, such as Bonnans and Ramírez [4] and in Bonnans and Shapiro's monograph [5] C^2-*cone-reducible* sets, such as the cone of symmetric positive semidefinite matrices and the second order cone (ice-cream cone or Lorentz cone) have been studied – especially strong regularity for linear and nonlinear SDP problems for C^2 functions.
Shapiro [46] summarizes an analysis of nonlinear semidefinite programs. He gives first-order necessary conditions for nonlinear programming under cone constraints. Furthermore, he shows necessary and sufficient second order con-

ditions under MF-condition (an extension of the Mangasarian-Fromowitz Constraint Qualifications (MFCQ)) and when the first order condition is given. He also looks at the geometry of the cone in SDP and shows that the transversality condition is a sufficient condition for uniqueness of the Lagrange multipliers. Shapiro [47] gives sufficient and sometimes necessary conditions for the uniqueness of the Lagrange multiplier, and hence of the solution of the dual problem in general Banach spaces. He discusses the differentiability properties of the optimal value and optimal solution of parametrized semidefinite programs in the convex and C^2 cases. In the special case of assuming unique and strictly complementary optimal solutions (under Slater's condition), Freund and Jarre [15] show a simple approach for the sensitivity of the perturbation of the solution of a linear SDP problem. They show that the solution functions of the perturbed SDP problem are differentiable, and that its derivatives are solutions of a system of linear equations. This, however, is not true for more general convex programs in conic form.

Several papers have supplied us with basic tools for SDP. Sun and Sun [49] calculate the directional derivative of the matrix absolute-value function and show that it, as well as the matrix projection function (onto the cone of positive semidefinite matrices) and the matrix projective residual function, is strongly semismooth. Chen, Qi, and Tseng [9] give an analysis on certain nonsmooth symmetric matrix-valued functions.

Both linear and nonlinear SDP has been looked at under strong regularity (cf. [8,16,48]). Assuming Robinson's constraint qualification (MFCQ for conventional nonlinear problems) D. Sun [48] proves that for a locally optimal solution, strong second order sufficient condition and constraint nondegeneracy are equivalent to nonsingularity of Clarke's Jacobian of the Karush-Kuhn-Tucker (KKT) system, and also equivalent to strong regularity of the KKT points. Chan and D. Sun [8] give more insightful characterizations for stationary points in the case of linear SDP problems, by proving that strong regularity is equivalent to nonsingularity of the B-subdifferential.

Fusek [18] shows that in nonlinear SDP strong and metric regularity are equivalent under certain conditions. For second-order cone programs, Outrata and D. Sun [41] describe the Aubin property by applying the limiting (Mordukhovich) coderivative, and Outrata and Ramírez [40] present the Aubin property in terms of a strong second-order optimality condition. However, their results have not been extended to semidefinite cones.

While both Clarke's generalized Jacobian and the B-subdifferential have been extended to SDP, there seem to be no results regarding the Thibault derivative in this field. The nonsingularity of the Thibault derivative of the KKT system characterizes sufficient and necessary conditions for stationary points for nonlinear SDP problems. Furthermore, we are not either aware of any results for the weak regularity condition *calmness* regarding SDP. In the current thesis, we try to bridge these gaps.

1.2 Current Thesis

Consider the following nonlinear optimization problem

$$\min_{x \in X} f(x) \text{ subject to } g(x) \in K \tag{1.1}$$

where $f\colon X \to \mathbb{R}$ and $g\colon X \to Y$ are continuously differentiable functions, X and Y are two finite-dimensional real vector spaces, each equipped with a scalar product denoted by $\langle .,. \rangle$ and K is a closed convex set in Y. In the following chapters, more specific and necessary restraints for the sets and functions given above are introduced.

We are interested in writing necessary optimality conditions of the above problem (1.1) as a model defined by a system of equations with a right-hand parameter p. Sensitivity analysis of problem (1.1) looks at disturbances of p in this model.

Solution methods of the problem (1.1) depend heavily on the definition of the set K. If the set K is a polyhedral set, e.g. cone of nonnegative real vectors, then the constraint can be rewritten as finitely many inequalities. Regarding K as a polyhedral set is a classical topic and a considerable amount of research has been done on sensitivity and stability analysis (cf. Robinson [44] and references therein and Section 1.1). However, for the nonpolyhedral case, such as the cone of negative semidefinite matrices, the applications from the polyhedral case are often not transferable.

Assume K is the set of negative (or positive) semidefinite matrices, then problem (1.1) is a semidefinite programming problem. In this case several advances in its analysis are currently being made (cf. Section 1.1). We wish to explore stability conditions for solution sets of SDP problems by extending and adapting applications and results from nonlinear complementarity problems to its semidefinite counterpart. This leads to new results both on generalized derivatives for the SDP case, and the extension of recent results on calmness to cone programs.

The main motivation behind this thesis is the sensitivity analysis of SDP problems by looking at the behavior of its solution sets, however, some results even apply to problems with more general sets K, such as general cones and convex sets.

In order to study critical points and solutions of the cone problem, we apply two different approaches. In the first approach, we look at generalized functions of the KKT conditions and work with generalized derivatives. In the second case, we study the convergence of an appropriate algorithm. These methods give us a sensitivity analysis of cone problems, more precisely, we show strong regularity, upper regularity, and calmness.

In our approach over generalized derivatives, we construct a Kojima-like locally Lipshitz function $\mathcal{F}(x, Y) = p$ of the KKT conditions for $C^{1,1}$- and C^2-optimization problems over the space of symmetric matrices. We study generalized derivatives of this Kojima function, in order to show regularity of our problem at $p = 0$. The Kojima function is the product of a continuous function dependent on $Df(x)$ and $Dg(x)$ and a nonsmooth function. The latter is nonsmooth due to the projection function onto the cone of positive semidefinite matrices. We look at the construction of its Thibault derivative (strict graphical derivative) and its contingent derivative (graphical derivative), and the resulting stability statements.

For our second approach, we are basically interested in calmness of a constraint system. For a multifunction F, such that $0 \in F(x)$ where $x \in K$, we get the parametrized model

$$\Sigma(p) := \{x \in K \mid p \in F(x)\}.$$

Assume $F(x)$ contains all p that fulfill $g(x) \leq p$ and rewrite $\Sigma(p) := \{x \in K \mid g(x) \leq p\}$. Here, $g\colon X \to \mathbb{R}^n$ is a continuously differentiable vector-valued function. Then, we arrive at the following optimization problem

$$\min_{x \in X} f(x) \text{ subject to } x \in \Sigma(0). \tag{1.2}$$

While problem (1.2) is slightly different from problem (1.1), it has the advantage of being written in the form of the KKT conditions. This enables us to even study calmness at stationary points for the entire optimization problem.

The set $\Sigma(p)$, the solution set of the above perturbed problem, is a mapping of a system of finitely many inequalities with perturbations on the right-hand side and a fixed algebraic constraint. Adding a fixed algebraic constraint to this parametrized model is new and is inspired by the structure of SDP problems. It enables us to consider an SDP problem in problem (1.2) by applying the cone of negative semidefinite matrices for the algebraic constraint K.

For this perturbed system, we study the calmness condition (cf. Klatte and Kummer [28] and references therein). We look at characteristics of calmness, the influence of the fixed algebraic constraint on the solution set, as well as a necessary and sufficient condition for this special multivalued mapping. Finally, we create the structure of an algorithm to show calmness.

2 Preliminaries

Looking at semidefinite programming problems means working with real symmetric matrices. However, in the calculus of symmetric matrices, we loose certain conditions and rules which are given in the vector case, as known from classical linear and nonlinear optimization. Furthermore, we encounter the hardship of algebraically representing the constraint of positive or negative semidefiniteness. For instance, we neither have polyhedral conditions nor the total ordering condition.
In this chapter, we give an overview and some basic characteristics of matrix analysis and SDP problems.

2.1 Semidefinite Programming (SDP)

Conic optimization is a general class of problems with variables or function values belonging to a cone. The classic optimization problems look at constraints over the nonpositive orthant $\mathbb{R}^n_-$, which is a cone in $\mathbb{R}^n$. The polyhedrality of this cone enables us to write the constraint conditions of an optimization problem as inequalities ($g_i(x) \leq 0$) and equalities. For the sake of simplicity, we omit the equality constraints.
Semidefinite programming problems are a special case of conic optimization problems. A matrix A is negative semidefinite if

$$x^T A x \leq 0 \text{ for all vectors } x \in \mathbb{R}^n, \tag{2.1}$$

and negative definite if the inequality is strict whenever x is nonzero. In the space $\mathcal{S}^n$ of symmetric $n \times n$ matrices, the cone of negative semidefinite matrices plays the analogous role to $\mathbb{R}^n_-$ above.
For points x and y in $\mathbb{R}^n$, we write $x \leq y$ if $x - y \in \mathbb{R}^n_-$ and $x < y$ if $x - y \in \mathbb{R}^n_{--}$ (the definition is analogous for $\geq$ and $>$). In the ordering of matrices X and Y in $\mathcal{S}^n$, the *Loewner partial ordering*, we write $X \preceq Y$ if $X - Y \in \mathcal{S}^n_-$ and $X \prec Y$ for $X - Y \in \mathcal{S}^n_{--}$ (the definition is analogous for $\succeq$ and $\succ$).
While both cones are convex, the major difference between these two cones is that the negative semidefinite cone is not a polyhedron (for $n > 1$). A

polyhedron can be defined with the help of finitely many linear inequalities, while a nonpolyhedral cone is described with polynomial inequalities (2.1) . For proofs in variational analysis, e.g. for the simplex method, the polyhedrality of the cone describing the constraints is used. It is unclear whether similar results apply to SDP.
In the case that the matrices are constrained to be diagonal, an SDP problem is reduced to a linear or nonlinear optimization problem over $\mathbb{R}^n$.

2.2 Notations

$\mathcal{S}^n$ is the linear space of all $n \times n$ symmetric real matrices and $\mathcal{S}^n_+ \subset \mathcal{S}^n$ ($\mathcal{S}^n_- \subset \mathcal{S}^n$) the cone of its positive (negative) semidefinite matrices. In a matrix $X \in \mathbb{R}^{n\times n}$, the element in the i-th row and j-th column is denoted as x_{ij}. Furthermore, we write

$$X := (x_{ij})_{\substack{i \in N \\ j \in N}} \text{ and } X_{IJ} := (x_{ij})_{\substack{i \in I \\ j \in J}}$$

where we have the index set $N = \{1, \ldots, n\}$ and I and J are subsets of N. We often just write $X = (x_{ij})_{NN}$. The dimension of an $n \times n$ matrix X is denoted as $\dim X = n$ and $\#N = n$ is the cardinality of N. For any two matrices $X, Y \in \mathcal{S}^n$, we have $\langle X, Y\rangle$ the standard inner product

$$\langle X, Y\rangle := \operatorname{tr} X^T Y = \sum_{ij} x_{ij} y_{ij},$$

where $\operatorname{tr} X := \sum_i x_{ii}$ is the trace of a matrix X and $X^T Y$ is the standard matrix multiplication. Note, that the transpose symbol can be omitted in the case of symmetric matrices. Furthermore, the operator $\circ$ gives us the Hadamard product $X \circ Y := (x_{ij} y_{ij})_{NN}$.

For a vector $z \in \mathbb{R}^n$, we denote the diagonal matrix Z with z on the main diagonal as $\operatorname{Diag} z := Z$; its dual $\operatorname{diag} Z := z$ writes the main diagonal of Z as a vector. The vector $\operatorname{vec} X$ is obtained by stacking all the columns of a matrix X. We easily see that $\operatorname{Diag} x \operatorname{Diag} y = \operatorname{Diag}(x \circ y)$ for $x, y \in \mathbb{R}^n$ and $\langle X, Y\rangle = \langle \operatorname{vec} X, \operatorname{vec} Y\rangle$.
The matrix $X \in \mathcal{S}^n$ has a spectral decomposition

$$X = P^T \Lambda P$$

where Λ is a diagonal matrix of the eigenvalues of X and P a corresponding orthogonal matrix. We have $PP^T = \mathbb{1}$ which is the identity matrix. The number of nonzero eigenvalues (counting multiplicities) is the rank of a matrix. Let $\mathcal{O}(X)$ be the set of all orthogonal matrices of a spectral decomposition of X, and let

$$\mathcal{SPEC}(X) := \{(P, \Lambda) \mid P \in \mathcal{O}(X), \Lambda = PXP^T\}$$

be the set of all tuples of the spectral decomposition of X.
The vector

$$\lambda := \operatorname{diag} \Lambda$$

contains all eigenvalues of X. We sort the eigenvalues with index sets $I_{(+)} := \{i \in \mathbb{N} \mid \lambda_i > 0\}$, $J := \{i \in \mathbb{N} \mid \lambda_i = 0\}$ and $I_{(-)} := \{i \in \mathbb{N} \mid \lambda_i < 0\}$. These sets do not have to yield a certain order.
The vector of eigenvalues of a symmetric matrix is a Lipschitz continuous function $\lambda(X)$ of the matrix entries. For a sequence X^k and given $(P^k, \Lambda^k) \in \mathcal{SPEC}(X^k)$, $\lambda_i^k(X^k)$ returns the i-th eigenvalue of X^k in relation to P^k. In this thesis the elements in the vector $\lambda(X)$ of all eigenvalues of X are not necessarily arranged in any certain order.
Let $\Lambda_{(+)} \in \mathcal{S}^p$ ($\Lambda_{(-)} \in \mathcal{S}^m$, and $\Lambda_{(0)} \in \mathcal{S}^z$) be a diagonal matrix that only contains the positive (negative and zero, respectively) eigenvalues of a $(p + m + z) \times (p + m + z)$ matrix, where $p, m, z \in \mathbb{N}$. These matrices are not to be confused with the diagonal matrices of all positive and, respectively, negative eigenvalues $\Lambda_+, \Lambda_- \in \mathcal{S}^n$ where $\Lambda = \Lambda_+ + \Lambda_-$.
The absolute value function $|.| : \mathcal{S}^n \to \mathcal{S}^n_+$ (cf. also Definition 2.3) with the following definition

$$|X| := \sqrt{X^2},$$

returns the matrix with the same eigenvectors as X and the eigenvalues given by the absolute value of the eigenvalues of X. For any $(P, \Lambda) \in \mathcal{SPEC}(X)$ we get

$$|X| = P^T \, |\Lambda| \, P.$$

The projection of a symmetric matrix onto the cone of positive semidefinite matrices is well defined (cf. [22, Theorem 2.1]), by replacing the negative eigenvalues of the concerning matrix with zero, and we denote

$$X_+ := P^T \operatorname{Diag}(\max\{\lambda_1, 0\}, \ldots, \max\{\lambda_n, 0\}) \, P = P^T \Lambda_+ P.$$

The same approach, by replacing the positive eigenvalues with zeros, leads to the projection onto the cone of negative semidefinite matrices, and clearly $X_+ + X_- = X$. We easily see that

$$X_+ = |X| + X_- = \frac{1}{2}(|X| + X). \tag{2.2}$$

The definitions above are independent of the choice of $(P, \Lambda) \in \mathcal{SPEC}(X)$.
Furthermore, we use the Lyapunov operator,

$$L_C(Z) := CZ + ZC, \quad Z \in \mathcal{S}^m$$

and its inverse operator L_C^{-1}, if it exists.
Finally, let us take a closer look at the differential of matrix-valued functions.

Definition 2.1. Let $F : \mathcal{X} \to \mathcal{Y}$, where $\mathcal{X}, \mathcal{Y}$ are matrix spaces, and $X, H \in \mathcal{X}$. If $DF : \mathcal{X} \to \mathcal{Y}$ is a linear operator that satisfies

$$\lim_{H \to 0} \frac{\|F(X + H) - F(X) - DF(X)H\|}{\|H\|} = 0,$$

then F is said to be *F(réchet)-differentiable* at X and $DF(x)$ is the F-derivative of F at X.

For a mapping $f\colon \mathbb{R}^n \to \mathbb{R}$, the F-derivative $Df(x)$ is written as a column vector

$$Df(x) = \begin{pmatrix} D_{x_1} f(x) \\ \vdots \\ D_{x_n} f(x) \end{pmatrix},$$

where $D_{x_i} f(x)$ is the partial derivative with respect to x_i. For a matrix-valued mapping $F\colon \mathbb{R}^n \to \mathcal{S}^p$, with $p \in \mathbb{R}$, the partial derivative with respect to x_i is the matrix

$$D_{x_i} F(x) = \begin{pmatrix} D_{x_i} F_{11}(x) & \dots & D_{x_i} F_{1p}(x) \\ \vdots & & \vdots \\ D_{x_i} F_{p1}(x) & \dots & D_{x_i} F_{pp}(x) \end{pmatrix}.$$

For $h \in \mathbb{R}^n$, we have $\operatorname{vec}(DF(x)h) = \langle D(\operatorname{vec} F(x)^T), h\rangle$, and the F-derivative $DF(x)$ in a direction h is

$$DF(x)h = \sum_{i=1}^{n} D_{x_i} F(x) h_i.$$

Let us now assume $F\colon \mathcal{S}^p \to \mathcal{S}^p$, then the F-derivative $DF(X)$ in a direction $H \in \mathcal{S}^p$ is

$$DF(X)H = \sum_{i,j=1}^{n} D_{x_{ij}} F(X) H_{ij},$$

where $\operatorname{vec}(DF(X)H) = \langle \operatorname{vec} D(\operatorname{vec} F(X)^T), \operatorname{vec} H\rangle$.
For a block-matrix with F-differentiable matrix-valued mappings

$$A(x) = \begin{pmatrix} A_{11}(x) & A_{12}(x) \\ A_{21}(x) & A_{22}(x) \end{pmatrix},$$

we write

$$DA(x) = \begin{pmatrix} DA_{11}(x) & DA_{12}(x) \\ DA_{21}(x) & DA_{22}(x) \end{pmatrix}$$

for the F-derivative. Considering a second F-differentiable block matrix mapping $B(y)$ and directions h, u, then the product rule is

$$D(A(x)B(y))(u,h) = [DA(x)u]B(y) + A(x)[DB(y)h].$$

There are similar result for the contingent derivative $CA(x)(u), CB(y)(h)$ and Thibault derivative $TA(x)(u), TB(y)(h)$, which we look at in detail in Chapter 3.

2.3 Toolbox

While working with SDP problems, a basic knowledge of matrix analysis is useful. In this section, a few characteristics from this field are reviewed, as well as some basic lemmata. Furthermore, a few interesting counterexamples are presented.

First of all, let A be a symmetric matrix, hence $A = A^T$, where A^T is the transposed matrix. We have the spectral decomposition $A = P\Lambda P^T$ with $(P, \Lambda) \in \mathcal{SPEC}(A)$ and the Euclidean norm

$$\|A\| = \sqrt{tr(A^2)} = \sqrt{\sum_{i\in N} \lambda^2(A)} = \sqrt{\sum_{i\in N}\sum_{j\in N} a_{ij}^2} = \sqrt{\sum_{i\in N} (a^2)_{ii}}.$$

By multiplying all the eigenvalues of A with each other, we get the determinant

$$\det A = \prod_{i\in N} \lambda_i(A),$$

while from the sum of the eigenvalues, we have the trace

$$\operatorname{tr} A = \sum_{i\in N} \lambda_i(A) = \sum_{i\in N} a_{ii}.$$

The eigenvalues are closely related to the diagonal of a matrix. It follows directly, that for $H \in \mathcal{S}^n$ we have

$$\sum_{i\in N} \lambda_i(H) = \sum_{i\in N} \lambda_i(\operatorname{Diag}(h_{ii})_{i\in N}).$$

However, we do not have $\|H\| = \|\operatorname{Diag}((h_{ii})_{i\in N})\|$, as can easily be seen in Example 2.2.

Example 2.2. The matrix

$$H = \begin{pmatrix} 1 & -1 \\ 2 & 4 \end{pmatrix}$$

has the eigenvalues $\lambda_1 = 3$ and $\lambda_2 = 2$. Then we get $\sum \lambda_i^2(H) = 13$ but $\sum \lambda_i^2(\operatorname{Diag}(1,4)^T) = 17$.

By calculating the spectral decomposition of A, we get its following description

$$A = \left(\sum_{s=1}^{n} p_{is}p_{js}\lambda_s\right)_{i\in N j\in N}, \tag{2.3}$$

where p_{ij} are the entries of P and $\lambda_s := \lambda_s(A)$. In Chapter 3, we put effort into the absolute value function of symmetric matrices, which is defined with help of the square root of a positive semidefinite matrix.

Let $A \in \mathcal{S}^n$ be positive semidefinite. Then, there exists a unique symmetric positive semidefinite matrix $B \in \mathcal{S}^n$ such that $B^2 = A$. We call B the square root of A and denote it by $B = \sqrt{A}$. Note, that $A^2 = AA^T$.

Definition 2.3. The matrix absolute value of a matrix $X \in \mathcal{S}^n$ is defined as $|X| := \sqrt{X^2}$.

For this function, many conclusions seem to follow automatically from the absolute value function for vectors. However, in several cases an extension to matrix calculus can lead to incorrect assumptions. In Lemma 2.4, we list useful tools for working with the absolute value of symmetric matrices.

Lemma 2.4. *Let $A, B \in \mathcal{S}^n$, $X \in \mathbb{R}^{n\times n}$, and $D = \operatorname{Diag} d$ for $d = (d_1 \dots d_n)^T \in \mathbb{R}^n$. Then we get the following statements:*

(i) $|X| \succeq 0$ *and* $|X| \in \mathcal{S}^n$.

(ii) $|X|^2 = (X)^2$.

(iii) $|A|\,|D| + |D|\,|A| = (|d_i| + |d_j|)_{NN} \circ |A| \in \mathcal{S}^n$.

(iv) $AD + DA = (d_i + d_j)_{NN} \circ A \in \mathcal{S}^n$.

(v) $|A|\,|B| = |AB|$ *if* $\mathcal{O}(A) \cap \mathcal{O}(B) \neq \emptyset$.

(vi) $|A|\,|B| + |B|\,|A| \in \mathcal{S}^n$.

(vii) $|AB| + |BA| \in \mathcal{S}^n$.

(viii) (Schur product theorem[1]) If $A, B \succeq 0$ *then* $A \circ B \succeq 0$.

(ix) $\left|PAP^T\right| = P\,|A|\,P^T$ *for any orthogonal matrix* P.

(x) Suppose $A, B \succeq 0$*, then we get the following equivalence:*

$$\langle A, B\rangle = 0 \iff AB = 0.$$

(xi) $A = (A+B)_+ \iff B = (A+B)_-$
$\iff A \succeq 0, B \preceq 0, \langle A, B\rangle = 0.$

Proof. (i) Let $X \in \mathbb{R}^{n\times n}$, then $|X| = \sqrt{XX^T}$, where $XX^T \in \mathcal{S}^n$ and $XX^T \succeq 0$. Then there exists a unique $Y \succeq 0$ such that $Y^2 = XX^T$, which fulfills $Y = \sqrt{XX^T} = |X|$.

[1] Also known as Hadamard's theorem.

(ii) By definition, we have $|X|^2 = \sqrt{XX^T}^2$. Since

$$vXX^Tv^T = vX(vX)^T \geq 0 \quad \forall v \in \mathbb{R}^n,$$

we have $XX^T \succeq 0$ and hence $|X|^2 = XX^T = X^2$.

(iii) The statement follows automatically by expansion and calculation. Let $A = P\Lambda P^T$ be a spectral decomposition of A, where $P = (p_{ij})_{NN}$, $N = \{1, \ldots, n\}$, and λ_i are the eigenvalues of the diagonal matrix Λ. Then, with (2.3), we have

$$|D|\,|A| = \begin{pmatrix} |d_1| \sum_s p_{1s}p_{1s} |\lambda_s| & \ldots & |d_1| \sum_s p_{1s}p_{ns} |\lambda_s| \\ \vdots & & \vdots \\ |d_n| \sum_s p_{ns}p_{1s} |\lambda_s| & \ldots & |d_n| \sum_s p_{ns}p_{ns} |\lambda_s| \end{pmatrix}$$

and

$$|A|\,|D| = \begin{pmatrix} |d_1| \sum_s p_{1s}p_{1s} |\lambda_s| & \ldots & |d_n| \sum_s p_{1s}p_{ns} |\lambda_s| \\ \vdots & & \vdots \\ |d_1| \sum_s p_{ns}p_{1s} |\lambda_s| & \ldots & |d_n| \sum_s p_{ns}p_{ns} |\lambda_s| \end{pmatrix},$$

which gives us the Hadamard product $|A|\,|D| + |D|\,|A| = (|d_i| + |d_j|)_{NN} \circ |A|$.

(iv) The proof is similar to the proof of *(iii)* above.

(v) W.l.o.g., we choose $P \in \mathcal{O}(A) \cap \mathcal{O}(B)$ then

$$AB = P\Lambda(A)\Lambda(B)P^T,$$

where $(P, \Lambda(A)) \in \mathcal{SPEC}(A)$ and $(P, \Lambda(B)) \in \mathcal{SPEC}(B)$. Then, we get

$$\begin{aligned} |AB| =& P\,|\Lambda(A)\Lambda(B)|\,P^T = P\,|\Lambda(A)|\,|\Lambda(B)|\,P^T \\ =& P\,|\Lambda(A)|\,P^T P\,|\Lambda(B)|\,P^T = |A|\,|B|\,. \end{aligned}$$

(vi) Since $(|A|\,|B|)^T = |B|\,|A|$ and $|A|\,|B| + (|A|\,|B|)^T \in \mathcal{S}^n$, we have $|A|\,|B| + |B|\,|A| \in \mathcal{S}^n$.

(vii) Since $|AB|^T = |BA|$ and $|AB| + |AB|^T \in \mathcal{S}^n$, we have $|AB| + |BA| \in \mathcal{S}^n$.

(viii) See e.g. the proof by Bernstein [3, Fact 8.22.12] for this well-known theorem by I. Schur[2].

(ix) For $(Q, \Lambda) \in \mathcal{SPEC}(A)$ and any orthogonal matrix P, we get

$$\left|PAP^T\right| = \left|PQ\Lambda Q^T P^T\right|.$$

Since PQ is an orthogonal matrix, it follows from the definition of the absolute value function and the spectral decomposition that

$$\left|PQ\Lambda Q^T P^T\right| = PQ\left|\Lambda\right| Q^T P^T = P\left|Q\Lambda Q^T\right| P^T = P\left|A\right| P^T.$$

(x) For $A, B \succeq 0$ there exist matrices C, D such that $A = CC^T$ and $B = DD^T$. Since $\operatorname{tr}(XY) = \operatorname{tr}(YX)$ for any $X, Y \in \mathcal{S}^n$, we can rewrite the trace of AB as

$$\langle A, B\rangle = \operatorname{tr}(CC^T DD^T) = \operatorname{tr}(D^T CC^T D) = \sum_{i=1}^{n}\sum_{j=1}^{n}[(C^T D)_{ij}]^2.$$

Hence, if $\operatorname{tr}(AB) = 0$ then we get $C^T D = 0$, and

$$AB = CC^T DD^T = 0.$$

The inverse direction, that $\langle A, B\rangle$ follows from $AB = 0$ is trivial, hence, the equivalence is proven.

(xi) Suppose, we have $A = (A + B)_+$ then we can write

$$(A + B)_- = A + B - (A + B)_+ = B.$$

Evidently, this is also true in the inverse direction, and the first equivalence follows.
Now, let us suppose $A = (A + B)_+$ and $B = (A + B)_-$, then, clearly, we have $A \succeq 0$, $B \preceq 0$ and

$$\langle A, B\rangle = \langle (A + B)_+, (A + B)_-\rangle = 0,$$

which is a sufficient condition for the last equivalence. For the necessary condition (the 'left-arrow' direction) consider $A, -B \succeq 0$ and $-\langle A, B\rangle =$

[2] Also known as J. Schur

$-\sum_i \lambda_i(AB) = 0$. As proven in (x), it follows that $AB = 0$, and the two matrices are simultaneously diagonalizable and have the same eigenvectors. Hence, we have the spectral decomposition

$$A^T B = P\Lambda P^T = P(\Lambda_A \Lambda_B)P^T = P\Lambda_A P^T P\Lambda_B P^T = 0,$$

where $\Lambda_A \Lambda_B = 0$ and the matrices Λ_A and Λ_B are perpendicular to each other. Since $A \succeq 0$, we get $\Lambda_A \succeq 0$ and $(A+B)_+ = P\Lambda_A P^T = A$. The same result follows for B and the equivalence is proven.

□

The downside of working with matrices is, that we loose some nice calculation laws that apply for vectors. This makes working with matrices more difficult and several results, that appear to be trivial, quite complicated. A few of these nontrivialities are listed in the following lemma.

Lemma 2.5. *In general, the following statements are not true for matrices* $A, B \in \mathcal{S}^n$*:*

(i) If $A \succeq 0$ *and* B *is component-wise positive, hence,* $b_{ij} \geq 0$ *for all* i, j*, then* $A \circ B \succeq 0$.

(ii) If $A, B \succeq 0$ *then* $AB + BA \succeq 0$.

(iii) $|A + B| \preceq |A| + |B|$.

(iv) (Not a lattice ordering) There exists a matrix $C \in \mathcal{S}^n$ *such that for all* $X \in \mathcal{S}^n$*, we have the following equivalence*

$$X \succeq A \text{ and } X \succeq B \iff X \succeq C.$$

Proof. (i) The statement can be disproved with the counterexample

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \text{ and } B = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix},$$

since in this case $A \circ B$ is indefinite.

(ii) The statement can be disproved with the same counterexample as in (i).

(iii) This statement was originally pointed out by Nemirovski, and the following counterexample is given by Jarre [53, Chapter 2.3]. Let $u = (1,0)^T$ and $v = (1,\varepsilon)^T$ for some small positive ε. Define $A = uu^T$ and $B = -vv^T$. Then

$$|A| + |B| = \begin{pmatrix} 2 & \varepsilon \\ \varepsilon & \varepsilon^2 \end{pmatrix} \text{ and } A + B = -\begin{pmatrix} 0 & \varepsilon \\ \varepsilon & \varepsilon^2 \end{pmatrix}.$$

The eigenvalues of $|A| + |B|$ are approximately 2 and $\frac{\varepsilon^2}{2}$, while the eigenvalues of $A + B$ are approximately $\pm\varepsilon$. Hence, $|A + B| \approx \varepsilon\mathbb{1}$, the eigenvalues of $|A + B|$ are about $\frac{2}{\varepsilon}$-times larger than the smallest eigenvalue of $|A| + |B|$.

(iv) The following counterexample is from Borwein and Lewis [6, Chapter 1.2]. Consider the matrices

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \text{ and } B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

For $A = \mathbb{1}$, the matrix C must be $\mathbb{1}$, too. However, for

$$A = \frac{2}{3}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix},$$

the choice of $C = \mathbb{1}$ contradicts the assumption.

□

Finally, we need the following lemma by Tseng [52, Lemma 6.2] to construct the Thibault limiting set of the absolute value function in Chapter 3.3.

Lemma 2.6. *Fix any $C \in \mathcal{S}^n_+$ and any orthogonal matrix P such that*

$$PCP^T = \begin{pmatrix} \tilde{C}_{II} & 0 \\ 0 & 0 \end{pmatrix}$$

holds for some $I \subset \{1, \dots, n\}$ and some positive definite submatrix $\tilde{C}_{II}$. For each $W \in \mathcal{S}^n$ with $C^2 + W \in \mathcal{S}^n_+$, upon letting $Z := (C^2 + W)^{\frac{1}{2}} - C$ and

$$\tilde{W} = \begin{pmatrix} \tilde{W}_{II} & \tilde{W}_{IJ} \\ \tilde{W}^T_{IJ} & \tilde{W}_{JJ} \end{pmatrix} := PWP^T, \tilde{Z} = \begin{pmatrix} \tilde{Z}_{II} & \tilde{Z}_{IJ} \\ \tilde{Z}^T_{IJ} & \tilde{Z}_{JJ} \end{pmatrix} := PZP^T,$$

where $J := \{1, \ldots, n\} \setminus I$, *we have*

$$\begin{aligned} \left\| \tilde{Z}_{JJ} \right\| &\leq n^{\frac{1}{4}} \left\| \tilde{W}_{JJ} \right\|^{\frac{1}{2}}, \\ \tilde{Z}_{IJ} &= \tilde{C}_{II}^{-1} \tilde{W}_{IJ} + o(\|W\|), \\ \tilde{Z}_{II} &= L_{\tilde{C}_{II}}^{-1} [\tilde{W}_{II}] + o(\|W\|). \end{aligned}$$

3 | Generalized Derivatives

In this chapter, we look at the generalized derivatives to characterize regularity conditions for SDP problems and the calculus thereof. Furthermore, we construct the parametrized model $\mathcal{F}(x, Y) = p$, a Kojima-like locally Lipschitz function of the Karush-Kuhn-Tucker (KKT) conditions for $C^{1,1}$- and C^2-optimization problems of SDP. The disturbances of the right-hand parameter p is analyzed with the help of generalized derivatives and enables us to study critical points and solutions of SDP problems.
We use the contingent derivative (graphical derivative) to study upper regularity, and the Thibault limit set (strict graphical derivative or Thibault derivative) to study strong regularity, respectively, of this Kojima function at a solution. Furthermore, we look at the Clarke generalized Jacobian in relation to the Thibault limit set.
Special attention is given to the construction of the concerned generalized derivative of the projection function onto the cone of positive semidefinite matrices, which is an important component of this SDP Kojima function.

First, a short overview of existing results concerning Kojima systems and generalized derivatives is given, and our results are embedded therein.
The structure of the current chapter is as following: After giving a motivation in Section 3.1, we look at the definition of upper regularity and the construction of the contingent derivative for the projection function on to the $\mathcal{S}^p_+$-cone in Section 3.2. In Section 3.3 we look at strong regularity and construct the Thibault limit set of the projection. In the Sections 3.1.2, 3.2.3 and 3.3.3 we explore the Kojima system for SDP and the conditions needed to show regularity. Finally, in Section 3.4 we give a short overview on the Clarke generalized Jacobian of the projection mapping and strong regularity.

3.1 Motivation and basic settings

In this subsection, we give our motivation for looking at generalized derivatives and the framework for the construction and application thereof. We give a short overview of the state of the art and later describe the Kojima system for SDP.

3.1.1 Motivation

Analysis of Lipschitz functions and their generalized derivatives

In the current chapter, we look at an analysis of Lipschitz functions and a selection of their generalized derivatives, to show certain regularity conditions. There are classical answers for solving optimization problems with sufficiently smooth functions and even certain nonsmooth ones, by defining reasonable derivatives and working with inverse function theorems. In the case of classical nonlinear programing over polyhedral sets, the application of calculus of generalized derivatives can be found in the work by Rockafellar and Wets [45], Klatte and Kummer [26], and references therein.
However, we are interested in sensitivity and stability analysis of solution sets of nonlinear optimization problems under set or cone constraints. We wish to extend this calculus to the case of nonpolyhedral cones such as the cone of positive or negative semidefinite matrices.
Bonnans and Ramírez [4] look at nonpolyhedral cones and give a characterization of strong regularity for nonlinear second order cone[1] programming (SOCP) problems (in terms of second order conditions), and Outrata and Ramírez [40] characterize Aubin property in terms of a strong second-order sufficient condition for nonlinear SOCP problems. show strong regularity in terms of a strong second-order optimality condition. However, this does not apply to SDP problems. Strong regularity of the KKT point for a locally optimal solution to the nonlinear SDP (NLSDP) problem has been given by D. Sun [48], but for stationary points, Chan and D. Sun [8] only refer to linear SDP problems. Freund, Jarre, and Vogelbusch [16] characterize stability for a nonlinear SDP problem, however, only the constraint mappings are nonlinear, and they refer to local solutions and assume local uniqueness of the KKT points. For all these regularity characterizations it has been assumed, that the critical points are also local minima of NLSDP problems.
We are interested in strong regularity of a KKT point for stationary points of NLSDP problems, without requiring that the points are local extrema.
We look at the basic tools for working with the contingent derivative and

[1] Second order cone:= $\{s \in \mathbb{R}^{n+1} \mid s_0 \geq \|(s_1, \ldots, s_n)\|\}$

Thibault limit set such as chain rules, where the involved problem-functions are not necessarily twice differentiable, and the derivatives of the projection function onto a cone. The chain rules enable the computation of generalized derivatives. Often, composed generalized derivatives of functions or multifunctions do not fulfill the standard chain rules but only a weaker form thereof.
For a $C^{1,1}$-optimization problem in the classical case mentioned above, injectivity of the Clarke generalized Jacobian gives us strong regularity, however, the inverse direction does not follow in general. Kummer [32] shows that by replacing this generalized derivative with the more restrictive Thibault limit set, we get a sufficient and necessary condition for strong regularity. Hence, the Thibault limit set can be regarded as an essential tool for locally Lipschitz functions.
We are interested in applying this theory to the SDP case. The Clarke generalized Jacobian and the directional derivative for projection functions onto the cone of positive semidefinite matrices have already been analyzed, and the Thibault limit set is roughly between these two generalized derivatives. Meng, Sun and Zhao [38] use the Thibault derivative of the Kojima system of a conic program to characterize strong regularity, however, they restrict their results to sufficient conditions by studying Clarke's generalized Jacobian. Malick and Sendov [36] use tensors to give a detailed account on the Clarke generalized Jacobian of the projection onto the $\mathcal{S}_+^p$-cone, while Chan and D. Sun [8] give a more abstract description of the Clarke generalized Jacobian, based on the convex hull of the B-differential. The main difference between the B-differential in a certain direction and the Thibault limit set is, that in the Thibault case we also consider sequences converging towards the solution, where at points of the sequence the function of concern is not necessarily differentiable. This gives the Thibault limit set a construction, that includes the B-differential; the reverse, however, is not necessarily true.

We do not go into the relationship between strong regularity and metric regularity. For optimization problems over polyhedral cones, Kummer [33] proved that strong regularity and metric regularity of the Kojima function are equivalent, which is mainly due to the structure of the polyhedral cone, as shown by Dontchev and Rockafellar [11]. For SDP problems the situation is unclear, since the positive semidefinite cone of symmetric matrices is not polyhedral. In the convex case, Klatte and Kummer show equivalence [29]. For nonlinear SDP programs, Fusek [18] looks at this problem in more detail.

Kojima system

In order to study critical points and solutions in SDP, we construct parametrized models of locally Lipschitz functions of the KKT conditions for $C^{1,1}$- and C^2-optimization problems over the space of symmetric matrices. These functions are the Kojima function or Kojima system introduced by Kojima [30]. To show regularity of our problem, we study generalized derivatives of this Kojima function, which can be written as the product of a continuous function dependent of the F-derivative of the problem functions and a nonsmooth function. The nonsmooth function - which reflects the complementarity problem that arises from the KKT conditions - contains the projection function onto a cone. For SDP this is the projection onto the cone of positive (or negative) semidefinite matrices, a nonpolyhedral cone - Pang, Sun and Sun [42] give an extensive analysis on this projection mapping. As mentioned above, we look at the construction of its contingent derivative and Thibault limit set.
Similar approaches for programs with polyhedral cone constraints have been researched (cf. Klatte and Kummer [26, 31, 32]). Depending on what kind of regularity of the Kojima system we want to show, different generalized derivatives are used, such as the Clarke generalized Jacobian, the contingent derivative, and the Thibault limit set.

3.1.2 Kojima System

The Kojima system is an approach to rewrite KKT points and stationary points of an optimization problem as the zeros of a particular nonsmooth function, which is adapted from Kojima's form of the KKT conditions [30]. Generalized derivatives of this function enable us to characterize the regularity of its zeros and to study the stable behavior of these critical points in an analytical way.
As shown by Kummer [31,32], and Klatte and Kummer [26], and the references therein, the Kojima system is useful for showing strong regularity of locally Lipschitz mappings. Klatte and Kummer [26] rewrite the KKT conditions for feasible points of a nonlinear optimization problem (NLP) as the zeros of some nonsmooth function $\mathcal{F}$ sending $\mathbb{R}^d$, $d \in \mathbb{N}$, into itself. The function $\mathcal{F}$ can be reformulated as a product containing a projection function.
We rewrite the Kojima function for SDP over C^2- and $C^{1,1}$-functions as the product of two functions which are either dependent of the primal variable or

the Lagrange multiplier, respectively.
A similar approach to the Kojima function for SDP can be found in Fusek's work [18].
Consider a nonlinear semidefinite optimization problem (NLSDP)

$$\min f(x) \text{ s.t. } G(x) \in \mathcal{S}^p_-, \tag{3.1}$$

where the functions $f : \mathbb{R}^n \to \mathbb{R}$, $g : \mathbb{R}^n \to \mathcal{S}^p$ are continuously differentiable near some point of interest. For certain cases in this section, we need $C^{1,1}$-functions (differentiable functions with Lipschitz continuous derivatives). We write $Df(x) := (D_{x_1}f(x), \ldots, D_{x_n}f(x))^T$ for the F(réchet)-derivative of f at x, where $D_{x_i}f(x)$ is the partial derivative with respect to x_i. The F-derivative of matrix-valued functions and the following notations are covered in more detail in Chapter 2.2.
The KKT conditions, i.e. a first-order optimality condition for the NLSDP, are constructed with help of the *Lagrange function* $\mathcal{L}\colon \mathbb{R}^n \times \mathcal{S}^p \to \mathbb{R}$

$$\mathcal{L}(x, \Gamma) := f(x) + \langle \Gamma, G(x) \rangle$$

and are

$$D_x\mathcal{L}(x, \Gamma) = 0, \quad \Gamma \in \mathcal{N}_{\mathcal{S}^p_-}\left(G(x)\right), \tag{3.2}$$

where $\mathcal{N}_A(b) := \{x \mid \langle x, a - b \rangle \leq 0, \forall a \in A\}$ is the normal cone of a set A at a point $b \in A$. By definition of $\mathcal{N}$, the matrix $G(x)$ must be in $\mathcal{S}^p_-$. Any point (x, Γ) satisfying (3.2) is a *KKT point* of the problem (3.1), and x is its stationary point, respectively. The inclusion $\Gamma \in \mathcal{N}_{\mathcal{S}^p_-}(G(x))$ is clearly equivalent to

$$G(x) \preceq 0, \, \Gamma \succeq 0, \, \langle \Gamma, G(x) \rangle = 0.$$

By introducing $Y := \Gamma + G(x)$, the function $\tilde{\mathcal{F}}\colon \mathbb{R}^n \times \mathcal{S}^p \to \mathbb{R}^n \times \mathcal{S}^p$ can be assigned to (3.1) with the following components

$$\begin{aligned}
\tilde{\mathcal{F}}_1(x, Y) &= Df(x) + D_x(\langle Y_+, G(x) \rangle) \\
\tilde{\mathcal{F}}_2(x, Y) &= G(x) - Y_-.
\end{aligned}$$

This function $\tilde{\mathcal{F}}$ is called the *Kojima function* to the program (3.1). By placing $\tilde{\mathcal{F}}$ in a matrix, we get the *SDP Kojima function* $\mathcal{F}$.

Note that $\operatorname{vec} A$ for an $n \times m$ matrix A is the $nm \times 1$ vector obtained by stacking all columns of A and in this case $\operatorname{Diag}(\operatorname{vec} Y_+, \ldots, \operatorname{vec} Y_+)$ is the $np^2 \times n$ diagonal block-matrix. Then, $\mathcal{F}$ has the form

$$\begin{aligned}\mathcal{F}(x, Y) &= \begin{pmatrix} \operatorname{Diag}\left(Df(x) + D_x(\langle Y_+, G(x)\rangle)\right) & 0 \\ 0 & G(x) - Y_- \end{pmatrix} \\ &= M(x)N(Y),\end{aligned} \tag{3.3}$$

where

$$\operatorname{Diag}(D_x(\langle Y_+, G(x)\rangle)) = \operatorname{Diag}\begin{pmatrix} \langle \operatorname{vec} Y_+, D_{x_1} \operatorname{vec} G(x)\rangle \\ \vdots \\ \langle \operatorname{vec} Y_+, D_{x_n} \operatorname{vec} G(x)\rangle \end{pmatrix}.$$

The matrix M is an $(n+p) \times (n + np^2 + 2p)$ matrix and N an $(n + np^2 + 2p) \times (n+p)$ matrix with the following construction

$$\begin{aligned} M(x) &= \begin{pmatrix} \operatorname{Diag}(Df(x)) & \operatorname{Diag}(D(\operatorname{vec} G(x)^T)) & 0 & 0 \\ 0 & 0 & G(x) & -\mathbb{1} \end{pmatrix} \\ &= \begin{pmatrix} D_{x_1} f(x) & 0 & 0 & \operatorname{vec}(D_{x_1} G(x))^T & 0 & 0 & 0 & 0 \\ & \ddots & & & \ddots & & \vdots & \vdots \\ 0 & & 0 & 0 & & 0 & & \\ 0 & 0 & D_{x_n} f(x) & 0 & 0 & \operatorname{vec}(D_{x_n} G(x))^T & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & G(x) & -\mathbb{1} \end{pmatrix}\end{aligned}$$

and

$$N(Y) = \begin{pmatrix} \mathbb{1} & 0 \\ \operatorname{Diag}(\operatorname{vec} Y_+, \ldots, \operatorname{vec} Y_+) & 0 \\ 0 & \mathbb{1} \\ 0 & Y_- \end{pmatrix},$$

where $\mathbb{1}$ is an $n \times n$ matrix.
If $\mathcal{F}(x, Y) = 0$, which is equivalent to $\tilde{\mathcal{F}}(x, Y) = 0$, then we call (x, Y) a *critical point* of $\mathcal{F}$ and immediately see that

$$\begin{aligned} &(x, \Gamma) \text{ KKT point} && \Rightarrow \quad (x, \Gamma + G(x)) \text{ critical point of } \mathcal{F} \\ &(x, Y) \text{ critical point of } \mathcal{F} && \Rightarrow \quad (x, Y_+) \text{ KKT point.} \end{aligned}$$

$N(Y)$ can be rewritten as

$$N(Y) = \begin{pmatrix} \mathbb{1} & 0 \\ \operatorname{Diag}(\operatorname{vec} Y_+, \ldots, \operatorname{vec} Y_+) & 0 \\ 0 & \mathbb{1} \\ 0 & Y - Y_+ \end{pmatrix}.$$

Note, that Y_+, the projection onto $\mathcal{S}^p_+$, is the nonsmooth part of our function. To show different regularity conditions of $\mathcal{F}(x, Y)$, we look at generalized derivatives of $\mathcal{F}$. For $f, g \in C^2$, the mapping $M(x)$ is F-differentiable and any generalized derivative of $M(x)$ coincides with $DM(x)$. What remains to be examined are the generalized derivatives of $\mathcal{F}(x, Y)$ and especially $N(Y)$ or more precisely of Y_+, and the chain rule for these generalized derivatives.

3.2 Upper Regularity

We are interested in the regularity of a function F, to study the stable behavior of critical points in a solution set $S := F^{-1}$ of an optimization problem. Upper regularity of F is shown with an approach over generalized derivatives, in this case the contingent derivative. In the following section, we look at the definitions for upper regularity and its relation to the contingent derivative. Furthermore, we look at the contingent derivative and its application to the SDP Kojima function.

3.2.1 Definition

Before looking at the definition for upper regularity, we discuss certain local Lipschitz properties. For metric spaces X and Y, we indicate by the symbol $f \in C^{0,1}(X, Y)$, that f is a locally Lipschitz function, and say that f is locally Lipschitz (continuous) on some neighborhood $U(x^0)$ of x^0 with a Lipschitz constant $L > 0$ (and with rank L) if

$$\mathrm{d}(f(x'), f(x'')) \leq L\, \mathrm{d}(x', x'') \quad \forall x', x'' \in U(x^0),$$

where $\mathrm{d}(x_a, x_b)$ is the metric distance between two points and $\mathrm{dist}(x_a, A)$ the shortest distance between a point x_a and a set A. For Banach spaces X and Y, $f \in C^1(X, Y)$ ($f \in C^{1,1}(X, Y)$) indicates that f is a function with continuous (locally Lipschitz) first F-derivatives.

Let now F be a multivalued mapping $F : X \rightrightarrows Y$ for normed spaces X and Y. The inverse $S := F^{-1}$, which is in general a multivalued mapping, defines the set $S(y) := \{x \in X \mid y \in F(x)\}$ for $y \in Y$. Furthermore, $U := U(x^0)$ and $V := V(y^0)$ define neighborhoods of x^0 and y^0 in X and Y, respectively. The mapping S is said to be *lower semicontinuous (l.s.c.)* at (y^0, x^0), if $x^0 \in S(y^0)$ and $\mathrm{dist}(x^0, S(y)) \to 0$ for each sequence $y \to y^0$. The following somewhat stricter definition is a Lipschitz property.

Definition 3.1. S is called *Lipschitz lower semicontinuous (Lipschitz l.s.c.)* at (y^0, x^0) if for a neighborhood V we have

$$\operatorname{dist}(x^0, S(y)) \leq L\, \mathrm{d}_Y(y^0, y) \quad \forall y \in V.$$

In other words, the solution set $S(y^0)$, when y^0 is disturbed, does not shrink too fast in reference to $x^0 \in S(y^0)$.
Another Lipschitz property, *locally upper Lipschitz*, refers to the rate that a solution set expands.

Definition 3.2. S is called *locally upper Lipschitz (locally u.L.)* at (y^0, x^0) with rank $L > 0$ if L and neighborhoods U and V exist such that for $y \in V$ we have

$$x \in S(y) \cap U \Rightarrow \mathrm{d}_X(x, x^0) \leq L\, \mathrm{d}_Y(y, y^0).$$

In other words, the local solution set cannot grow too fast. Locally upper Lipschitz is often referred to as *isolated calmness*.
Now, let us look at upper regularity.

Definition 3.3. F is called *upper regular* at (x^0, y^0) if there exist $L > 0$ and neighborhoods U and V such that

$$\emptyset \neq F^{-1}(y) \cap U \subset B(x^0, L\, \mathrm{d}_Y(y, y^0)) \quad \forall y \in V.$$

We easily see that upper regularity of F requires that F^{-1} be Lipschitz l.s.c. at (y^0, x^0). Furthermore, if F^{-1} is locally upper Lipschitz at (y^0, x^0) and $F^{-1}(y) \cap U \neq \emptyset$ for all $y \in V$ then F is upper regular at (x^0, y^0). This leads to the following lemma, which was proven by Klatte and Kummer [26, Lemma 3.2] (cf. Lemma 3.7 below).

Lemma 3.4. *F is upper regular at (x^0, y^0) if and only if F^{-1} is locally u.L. and Lipschitz l.s.c. at (y^0, x^0).*

Next, we see that the contingent derivative can be used to show upper regularity.

3.2.2 Contingent Derivative

To get a picture of the contingent derivative, we first look at the *tangent cone* as defined by Rockafellar and Wets [45]. This cone is the graph of the contingent derivative.

Definition 3.5 (Tangent cone)**.** Let $C \subset X$ and $x^0 \in C$. The *tangent cone* $T_B(x^0, C)$ consists of all $w \in X$ such that for some $t \searrow 0$ and some $w_t \to w$ there holds $x^0 + tw_t \in C$. In other words, we have

$$\begin{aligned} T_B(x^0, C) &= \limsup_{t \searrow 0} t^{-1}(C - x^0) \\ &= \bigcap_{\varepsilon > 0} \bigcap_{\alpha > 0} \bigcup_{t \in]0, \alpha]} (t^{-1}(C - x^0) + \varepsilon B). \end{aligned}$$

The idea of the tangent cone dates back to Bouligand's contingent in 1932 [7], which is the set $x^0 + T_B(x^0, C)$, and is also called the Bouligand cone or the contingent cone (for more details refer to e.g. Aubin [2], Klatte and Kummer [26]). Rockafellar and Wets [45, p.198-199] give a nice geometrical interpretation of the tangent cone by describing it as a local approximation around x^0 through global magnification of C, if these magnifications converge to something. However, they also show that this geometric derivability is not always applicable and give the counterexample, where $C = \{(x_1, x_2) \mid x_1 \neq 0 \wedge x_2 = x_1 \sin(\frac{1}{x_1})\} \cup \{(0, 0)\}$ and look at the tangent cone $T_B((0, 0), C)$.
Let us now look at the *contingent derivative* of $F : X \rightrightarrows Y$, a multifunction between normed spaces as defined by Aubin and Ekeland [2]. Rockafellar and Wets [45] introduced it as the *graphical derivative*.

Definition 3.6 (Contingent derivative)**.** The mapping $CF(x, y) : X \rightrightarrows Y$ is called the *contingent derivative* and $v \in CF(x, y)(u)$ if there exist certain $t_k \searrow 0$ and $(x, y) \in \operatorname{graph} F$ and $(u^k, v^k) \to (u, v)$ such that $y + t_k v^k \in F(x + t_k u^k)$.
Assuming that F is a locally Lipschitz function with rank L near x, we rewrite the definition of the contingent derivative with

$$CF(x)(u) = \left\{ v \,\middle|\, \begin{array}{c} v = \lim t_k^{-1}[F(x + t_k u) - F(x)] \\ \text{for certain } t_k \searrow 0 \end{array} \right\}. \tag{3.4}$$

Consider F locally Lipschitz and $Y = \mathbb{R}^m$, then $CF(x)(u)$ is nonempty, closed, and bounded.
The graph of the contingent derivative is the tangent cone; hence,

$$\operatorname{graph} CF(x, y) = T_B\left((x, y), \operatorname{graph} F(x)\right).$$

We call the contingent derivative of F *injective* at (x, y) if $0 \notin CF(x, y)(u)$ for all $u \neq 0$.

The following lemma by Klatte and Kummer [26, Lemma 3.2], which is based on a result by King and Rockafellar [25], shows the connection between the contingent derivative of a function and its local properties.

Lemma 3.7 (King, Rockafellar 1992, Klatte, Kummer 2002). *Let F be the multivalued mapping $F : X \rightrightarrows Y$ between normed spaces, and let $(x,y) \in \operatorname{graph} F$. If F^{-1} is locally upper Lipschitz at (x,y) then $CF(x,y)$ is injective. If $X = \mathbb{R}^n$, then it holds*

$$CF(x,y) \text{ is injective } \iff F^{-1} \text{ is locally upper Lipschitz at } (y,x)$$

and

$$F \text{ is upper regular at } (x,y) \iff \begin{array}{c} CF(x,y) \text{ is injective and} \\ F^{-1} \text{ is Lipschitz l.s.c. at } (y,x). \end{array}$$

Referring to Lemma 3.7, we look at the contingent derivative of the Kojima function to show upper regularity.

Corollary 3.8. *The Kojima function $\tilde{\mathcal{F}}$ is upper regular at $((x^0,Y^0),z^0)$ if and only if $C\tilde{\mathcal{F}}(x^0,Y^0)$ is injective and $\tilde{\mathcal{F}}^{-1}$ is Lipschitz l.s.c. at $(z^0,(x^0,Y^0))$.*

Proof. From Lemma 3.4, we know that $\tilde{\mathcal{F}}$ is upper regular at $((x^0,Y^0),z^0)$ iff $\tilde{\mathcal{F}}^{-1}$ is locally u.L. and Lipschitz l.s.c. at $(z^0,(x^0,Y^0))$.
What remains to be shown is that locally upper Lipschitz continuity of $\tilde{\mathcal{F}}^{-1}$ at $(z^0,(x^0,Y^0))$ is equivalent to injectivity of $C\tilde{\mathcal{F}}(x^0,Y^0)$. In Lemma 3.7 the sufficient condition for injectivity is proven, and the equivalence is given when the domain is $\mathbb{R}^n$. However, the proof in Lemma 3.7 is also valid for $\tilde{\mathcal{F}}$. For consistency, we give the proof for the necessary condition.
Suppose, the locally upper Lipschitz condition cannot be satisfied for each choice of U, V, and L, this means that there are sequences

$$\begin{gathered} ((x^k,Y^k),z^k) \to ((x^0,Y^0),z^0) \text{ in } \operatorname{graph} \tilde{\mathcal{F}} \text{ for } k \to \infty \\ \text{such that } t_k := \mathrm{d}\left((x^k,Y^k),(x^0,Y^0)\right) > k\,\mathrm{d}(z^k,z^0). \end{gathered}$$

The quotients $v^k := \frac{1}{t_k}\mathrm{d}(z^k,z^0)$ are vanishing and the bounded sequence $u^k := \frac{1}{t_k}((x^k,Y^k)-(x^0,Y^0))$ has an accumulation point u. Hence, the limit of v^k can be written by means of the contingent derivative of $\tilde{\mathcal{F}}$ as

$$0 \in \tilde{\mathcal{F}}(x^0,Y^0)(u) \text{ for some } u \neq 0.$$

This concludes the proof. ☐

The following Lemma 3.9 enables us to work with the SDP Kojima system.

Lemma 3.9. *We have injectivity for $C\tilde{\mathcal{F}}$ at (x, Y) if and only if we have injectivity for $C\mathcal{F}$ at (x, Y).*

Proof. Assume $C\tilde{\mathcal{F}}$ is injective at (x, Y), then if $0 \in C\tilde{\mathcal{F}}(x, Y)(u, H)$ we only have the trivial solution $(u, H) = 0$.
Injectivity of $C\tilde{\mathcal{F}}$ at (x, Y) is equivalent to writing, that if

$$\begin{aligned} &0 \in C\left(Df(x) + D_x\langle Y_+, G(x)\rangle\right)(u, H) \text{ and} \\ &0 \in C(G(x) - Y_-)(u, H), \end{aligned}$$

then $(u, H) = 0$. We have

$$\begin{aligned} &C\left(Df(x) + D_x\langle Y_+, G(x)\rangle\right)(u, H) = \\ &C_x\left(Df(x) + D_x\langle Y_+, G(x)\rangle\right)u + C_Y\left(D_x\langle Y_+, G(x)\rangle\right)H \end{aligned}$$

and

$$\begin{aligned} C(G(x) - Y_-)(u, H) &= C(G(x) - Y + Y_+)(u, H) \\ &= DG(x)u - H + C\Pi_+(Y)H. \end{aligned}$$

Whence the reformulation of the assumption above, that if

$$\begin{aligned} &0 \in C\mathcal{F}(x, Y)(u, H) \\ &\quad = (C_x\mathcal{F}(x, Y), C_Y\mathcal{F}(x, Y))(u, H) \\ &\quad = \begin{pmatrix} \mathrm{Diag}(C_x\left(Df(x) + D_x\langle Y_+, G(x)\rangle\right)u + C_Y\left(D_x\langle Y_+, G(x)\rangle\right)H) & 0 \\ 0 & DG(x)u - H + C\Pi_+(Y)H \end{pmatrix}, \end{aligned}$$

then $(u, H) = 0$; hence, $\mathcal{F}$ is injective at (x, Y). $\square$

Finally, from Corollary 3.8 and Lemma 3.9 we get Corollary 3.10 to characterize upper regularity with the help of the SDP Kojima system.

Corollary 3.10. *$\tilde{\mathcal{F}}$ is upper regular at $((x^0, Y^0), z^0)$ if and only if $C\mathcal{F}(x^0, Y^0)$ is injective and $\tilde{\mathcal{F}}^{-1}$ is Lipschitz l.s.c. at $(z^0, (x^0, Y^0))$.*

Directional Derivative for the Projection Function Π_+

A locally Lipschitz function F is said to be *directionally differentiable* if

$$F'(x)(u) = \lim_{t \searrow 0} \frac{F(x+tu) - F(x)}{t}$$

exists. Consider F a locally Lipschitz function having directional derivatives. Then, the choice of sequences t_k as given in equation (3.4) for the contingent derivative is arbitrary and the contingent and directional derivatives coincide. As mentioned above, we are interested in the local properties of the SDP Kojima function, in particular the projection function.
We will note

$$\Pi_+ : \mathcal{S}^n \to \mathcal{S}^n_+$$

as the projection function onto the cone of positive semidefinite matrices.
The projection function is covered in more detail in Chapter 2.2.
Note, that the projection function is directionally differentiable, so it suffices to look at the directional derivative to show injectivity of the contingent derivative. Sun and Sun [49, Theorem 4.7] give a formula for the directional derivative of the absolute value function $|Y|$ for $Y \in \mathcal{S}^n$, whence the projection function, since $\Pi_+(Y) = \frac{1}{2}[|Y| - Y]$.
We recall the notations in Chapter 2.2 and for $X \in \mathcal{S}^n$ consider $(P, \Lambda) \in \mathcal{SPEC}(X)$ such that

$$\Lambda = \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix},$$

where D is a nonsingular diagonal matrix and $C := |D|$. Let m denote the rank of X, and we have the index sets $K := \{1, \dots, m\}$ and $J := \{1, \dots, n\} \backslash K$. The inverse operator of the Lyapunov function L_C^{-1} is defined for matrices in $\mathcal{S}^m$. Furthermore, let $\tilde{H} := P^T H P$.

Theorem 3.11 (Sun, Sun 2002)**.** *Let $\Phi\colon \mathcal{S}^n \to \mathcal{S}^n$ be defined by $\Phi(Y) = |Y|$ for $Y \in \mathcal{S}^n$. Then, Φ is directionally differentiable at any $X \in \mathcal{S}^n$ and for any $H \in \mathcal{S}^n$, and the directional differential has the following form*

$$\Phi'(X)(H) = P \begin{pmatrix} L_C^{-1}[D\tilde{H}_{KK} + \tilde{H}_{KK} D] & C^{-1} D \tilde{H}_{KJ} \\ \tilde{H}_{KJ}^T D C^{-1} & \left|\tilde{H}_{JJ}\right| \end{pmatrix} P^T. \tag{3.5}$$

The existence of the directional derivative of Φ was already proven earlier by Bonnans et al. (cf. Bonnans and Shapiro [5] and references therein), and since $\Pi_+(Y) = \frac{1}{2}[|Y| + Y]$, we have

$$\Pi'_+(X)(H) = \frac{1}{2}[\Phi'(X)(H) + H]. \tag{3.6}$$

3.2.3 Upper Regularity of the Kojima System

In order to show upper regularity of the SDP Kojima system $\mathcal{F}$, the contingent derivative of $\mathcal{F}$ must be injective. We need a chain rule for the contingent derivative, to work with the product representation of the Kojima function $\mathcal{F}$, as shown in (3.3). This has been studied by Klatte and Kummer [26, Chapter 6.4.1] and leads to the following lemma. For completeness, we give a short proof.

Lemma 3.12. *Let $G : \mathcal{S}^m \to \mathcal{S}^n$ and $F : \mathcal{S}^n \to \mathcal{S}^p$ be locally Lipschitz, and we define $\Phi(\cdot) := F(G(\cdot))$ and $Y := G(X)$. Then, we have*

$$C\Phi(X)(U) \subset CF(Y)\left(CG(X)(U)\right) := \bigcup_{V \in CG(X)(U)} CF(Y)(V).$$

If F or G is directionally differentiable, then we get

$$C\Phi(X)(U) = CF(Y)\left(CG(X)(U)\right).$$

Proof. For any $W \in C\Phi(X)(U)$, we have

$$W = \lim_{t \searrow 0} t^{-1}\left[F\left(G(X+tU)\right) - F(G(X))\right].$$

We can choose a subsequence $V_t := t^{-1}[G(X+tU) - G(X)]$, hence $G(X+tU) = G(X) + tV_t$, with the limit $V \in CG(X)(U)$, since G is locally Lipschitz. Then, by replacing $G(X) = Y$, we get

$$W = \lim_{t \searrow 0} t^{-1}[F(Y + tV_t) - F(Y)],$$

and since F is locally Lipschitz, we get $W \in CF(Y)(V)$.
If F or G is directionally differentiable, then $V \in CG(X)(U)$ and $W \in CF(Y)(V)$ do not require different sequences $t \searrow 0$, and we get

$$C\Phi(X)(U) = CF(Y)\left(CG(X)(U)\right).$$

□

As a product rule, this yields Theorem 3.13 [26, Corollary 6.12].

Theorem 3.13. *(Klatte, Kummer 2002, Contingent product rule) Let $F(x,y) = M(x)N(y)$, where $M(\cdot)$ and $N(\cdot)$ are locally Lipschitz matrix-valued functions of related size. Suppose, that one of them is directionally differentiable. Then,*

$$CF(x,y)(u,v) = [CM(x)(u)]N(y) + M(x)[CN(y)(v)].$$

Since the projection function Π_+ is directionally differentiable, as mentioned above, so is $N(Y)$ of the SDP Kojima function. Furthermore, we have $M \in C^{0,1}$. Hence, we get

$$C\mathcal{F}(x,Y)(u,H) = [CM(x)(u)]\, N(Y) + M(x)[N'(Y)(H)]$$

and Theorem 3.14 on the SDP Kojima system.

Theorem 3.14. *The Kojima function $\tilde{\mathcal{F}}$ is upper regular at $((x,Y),0)$ if and only if $\tilde{\mathcal{F}}^{-1}$ is Lipschitz l.s.c. at $(0,(x,Y))$, and for $(u,H) \neq 0$ we have*

$$0 \notin C\mathcal{F}(x,Y)(u,H) = M(x)[N'(Y)(H)] + [CM(x)(u)]N(Y). \tag{3.7}$$

The statement in (3.7) is true if and only if for each solution of the system

$$\begin{aligned} &DG(x)u = H - \Pi_+'(Y)(H),\\ &0 \in D(\operatorname{vec} G(x)^T)\operatorname{vec}(\Pi_+'(Y)(H)) + C(Df(x) + D(\operatorname{vec} G(x)^T)\operatorname{vec} Y_+)(u) \end{aligned}$$

we have (u,H)=$(0,0)$, where

$$\Pi_+'(Y)(H) = \frac{1}{2}P\begin{pmatrix} L_C^{-1}[D\tilde{H}_{KK} + \tilde{H}_{KK}D] + \tilde{H}_{KK} & (C^{-1}D + \mathbb{1})\tilde{H}_{KJ} \\ \tilde{H}_{KJ}^T(DC^{-1} + \mathbb{1}) & 2\Pi(\tilde{H}_{JJ}) \end{pmatrix} P^T.$$

If $f, g \in C^2$, then we can write $DM(x)u$ instead of $CM(x)(u)$.

Proof. From Corollary 3.10, we know that $\tilde{\mathcal{F}}$ is upper regular at $((x^0,Y^0),0)$ if and only if $\tilde{\mathcal{F}}^{-1}$ is Lipschitz l.s.c. at $(0,(x^0,Y^0))$ and $C\mathcal{F}(x^0,Y^0)$ is injective; hence if 0 is included in the following equations for $C\mathcal{F}(x,Y)(u,H)$, then (u,H) must be 0. It remains to prove that the system stated below has only trivial solutions. For this, we compute the contingent derivative

$C\mathcal{F}(x,Y)(u,H)$. The equivalence follows from Theorem 3.13.

$$
\begin{aligned}
&C\mathcal{F}(x,Y)(u,H)\\
&=M(x)[N'(Y)(H)]+[CM(x)(u)]N(Y)\\
&=\begin{pmatrix}\mathrm{Diag}(Df(x)) & \mathrm{Diag}(D(\operatorname{vec}G(x)^T)) & 0 & 0\\ 0 & 0 & G(x) & -\mathbb{1}\end{pmatrix}\\
&\qquad\begin{pmatrix} 0 & 0\\ \mathrm{Diag}(\operatorname{vec}(\Pi'_+(Y)(H)),\dots,\operatorname{vec}(\Pi'_+(Y)(H))) & 0\\ 0 & 0\\ 0 & H-\Pi'_+(Y)(H)\end{pmatrix}\\
&\quad+\begin{pmatrix}\mathrm{Diag}\left(CDf(x)(u)\right) & \mathrm{Diag}\left(CD\big(\operatorname{vec}G(x)^T\big)^T(u)\right) & 0 & 0\\ 0 & 0 & DG(x)u & 0\end{pmatrix}\\
&\qquad\begin{pmatrix}\mathbb{1} & 0\\ \mathrm{Diag}\left(\operatorname{vec}Y_+,\dots,\operatorname{vec}Y_+\right) & 0\\ 0 & \mathbb{1}\\ 0 & Y_-\end{pmatrix}.
\end{aligned}
$$

We have

$$
\begin{aligned}
\mathrm{Diag}(D(\operatorname{vec}G(x)^T))\,&\mathrm{Diag}(\operatorname{vec}\Pi'_+(Y)(H),\dots,\operatorname{vec}\Pi'_+(Y)(H))\\
&=\mathrm{Diag}(D(\operatorname{vec}G(x)^T)\operatorname{vec}\Pi'_+(Y)(H))
\end{aligned}
$$

and

$$
\begin{aligned}
&\mathrm{Diag}\left(CDf(x)(u)\right)\\
&\quad+\mathrm{Diag}\left(CD\big(\operatorname{vec}G(x)^T\big)(u)\right)\mathrm{Diag}\left(\operatorname{vec}Y_+,\dots,\operatorname{vec}Y_+\right)\\
&\quad\;=\mathrm{Diag}(C(Df(x)+D(\operatorname{vec}G(x)^T)\operatorname{vec}Y_+)(u)).
\end{aligned}
$$

Whence, the following reformulation of $C\mathcal{F}(x,Y)(u,H)$

$$
C\mathcal{F}(x,Y)(u,H)=\begin{pmatrix}A & 0\\ 0 & DG(x)u-H+\Pi'_+(Y)(H)\end{pmatrix},
$$

where

$$
\begin{aligned}
A:=\mathrm{Diag}(D(\operatorname{vec}G(x)^T)&\operatorname{vec}\Pi'_+(Y)(H)\\
&+C(Df(x)+D(\operatorname{vec}G(x)^T)\operatorname{vec}Y_+)(u))
\end{aligned}
$$

and $0\in C\mathcal{F}(x,Y)(u,H)$ iff $0\in A$ and $DG(x)u=H-\Pi'_+(Y)(H)$. $\square$

3.3 Strong Regularity

A stricter regularity condition, that can also be described with generalized derivatives, is strong regularity. Strong regularity of a mapping F at x^0 comes to be identified with the property that F has pseudo regularity at x^0, and that the inverse mapping F^{-1} is (locally) uniquely defined. After exploring the properties of strong regularity, we take up its connection to the Thibault limit set.

3.3.1 Definition

Consider $S = F^{-1}$ the inverse of a given multifunction $F : X \rightrightarrows Y$, as defined above.
In order to describe *strong regularity*, we need the definition of the generally weaker *pseudo regularity*. This regularity is characterized by *pseudo Lipschitz*.

Definition 3.15 (Pseudo regularity)**.** S is called *pseudo Lipschitz* or has the *Aubin property* with rank L at (y^0, x^0) if there exists neighborhoods U and V of x^0 and y^0, respectively, such that for any $(y, x) \in (V \times U) \cap \operatorname{graph} S$ and $y' \in V$ there exists an $x' \in S(y')$ such that

$$d_X(x', x) \leq L \, \mathrm{d}_Y(y', y).$$

If S is pseudo Lipschitz at (y^0, x^0), then F is called *pseudo regular* at (x^0, y^0).

Obviously, a locally single-valued multifunction S is pseudo Lipschitz at x^0 if and only if it is Lipschitz continuous on some neighborhood of x^0.
The condition pseudo regularity states, that S has Lipschitz behavior near (y^0, x^0), respectively, that a Lipschitzian error estimate holds true locally around (x^0, y^0). The pseudo Lipschitz property was defined by Aubin and Ekeland [2], and the term Aubin property was introduced in Rockafellar and Wets [45].
A weaker regularity, where the Lipschitz behavior is only guaranteed at the solution point itself, is calmness, which is pursued in Chapter 4.
Provided with the definition of pseudo regularity, we can now define the condition strong regularity.

Definition 3.16 (Strong regularity)**.** For metric spaces X and Y, we call $F : X \to Y$ *strongly regular* at a point (x^0, y^0) if its inverse set-valued mapping F^{-1} is pseudo Lipschitz at (y^0, x^0) and there exist neighborhoods U and V of x^0 and y^0, respectively, such that $U \cap F^{-1}(y)$ is single valued for $y \in V$.

3.3.2 Thibault Derivative

The *Thibault limit set* or the *Thibault derivative* is a set valued mapping first considered by Thibault [50, 51] for Lipschitz functions. Rockafellar and Wets [45] introduce it as *strict graphical derivatives*. Our interest lies in the Thibault derivative for functions in real matrix spaces.

Definition 3.17 (Thibault derivative)**.** The mapping $TF(x, y) : X \rightrightarrows Y$ is called the *Thibault derivative* and $v \in TF(x, y)(u)$ if there exist certain $t_k \searrow 0$ and assigned points $(x^k, y^k) \in \operatorname{graph} F$ with $(x^k, y^k) \to (x, y)$ and $(u^k, v^k) \to (u, v)$ such that $y^k + t_k v^k \in F(x^k + t_k u^k)$.
Assuming that F is a locally Lipschitz function with rank L near x, we rewrite the definition of the Thibault derivative with

$$TF(x)(u) = \left\{ v \,\middle|\, \begin{matrix} v = \lim_{k\to\infty} t_k^{-1}[F(x^k + t_k u) - F(x^k)] \\ \text{for certain } t_k \searrow 0 \text{ and } x^k \to x \end{matrix} \right\}.$$

Consider F locally Lipschitz and $Y = \mathbb{R}^m$, then $TF(x)(u)$ is nonempty, closed, and bounded.
We call the Thibault derivative of F *injective* at a point (x, y) if we have $0 \notin TF(x, y)(u)$ for all $u \neq 0$.
The convex hull of the Thibault limit set gives us the Clarke generalized Jacobian, which we discuss in Section 3.4.
While for a continuously differentiable function $f : \mathbb{R}^n \to \mathbb{R}^n$, the contingent and the Thibault derivative coincide, hence $Cf = Tf = Df$, in general, we have $Tf \neq Cf$, as can be seen in the following example.

Example 3.18. Let $f(x) = |x|$. Then we get $Tf(0)(u) = [-|u|, |u|]$ but $Cf(0)(u) = f'(0; u) = |u|$, where $f'(x; u)$ is the usual directional derivative.

Condition for Strong Regularity

Kummer [31], and Klatte and Kummer [26, Theorem 5.14] show the connection between strong regularity of a locally Lipschitz function $F \in C(\mathbb{R}^n, \mathbb{R}^n)$ and injectivity of the Thibault derivative. In Theorem 3.19, we get a similar result for functions in $\mathcal{S}^p$.

Theorem 3.19. *A function $F \in C(\mathcal{S}^p, \mathcal{S}^p)$ is* strongly regular *at x^0 if and only if there exists a $c > 0$ such that*

$$\|F(x) - F(x')\| \geq c \|x - x'\| \quad \forall x, x' \in B(x^0, c). \tag{3.8}$$

Moreover, if $F \in C^{0,1}(\mathcal{S}^p, \mathcal{S}^p)$ and U is some neighborhood of X^0, then the following statements are equivalent:

(i) F is strongly regular at X^0.

(ii) $TF(X^0)$ is injective.

Proof. The proof closely follows the proof of Theorem 5.14 by Klatte and Kummer [26], since $\mathcal{S}^p$ is isomorphic to $\mathbb{R}^n$ (let $n := \frac{p^2+p}{2}$). □

Remark 3.20. It is not necessary to require Lipschitz continuity for F in Theorem 3.19.

Since we are interested in strong regularity of the SDP Kojima system, we need a condition to show its injectivity at a given point. Corollary 3.21 shows the equivalence of injectivity of $\tilde{\mathcal{F}}$ and $\mathcal{F}$ by replacing the contingent derivative with the Thibault derivative in the proof of Lemma 3.9.

Corollary 3.21. *We have injectivity for $T\tilde{\mathcal{F}}$ at (x, Y) if and only if we have injectivity for $T\mathcal{F}$ at (x, Y).*

Corollary 3.22, which follows directly from the proof of Theorem 3.19, results by Klatte and Kummer [26, Theorem 5.14], and Lemma 3.21 enables us to apply Theorem 3.19 to the Kojima functions $\mathcal{F}$ and $\tilde{\mathcal{F}}$.

Corollary 3.22. *Theorem 3.19 is also valid for a mapping F between $\mathbb{R}^n \times \mathcal{S}^p$ and $\mathbb{R}^n \times \mathcal{S}^p$.*

And finally we get:

Corollary 3.23. *Suppose $\tilde{\mathcal{F}} \in C^{(0,1)}(\mathbb{R}^n \times \mathcal{S}^p, \mathbb{R}^n \times \mathcal{S}^p)$ and U is some neighborhood of X^0, then:*
$\tilde{\mathcal{F}}$ is strongly regular at (x, Y) if and only if $\mathcal{F}$ is injective at (x, Y).

Our aim is to show stability of the SDP Kojima system by looking at its Thibault derivative, in particular the Thibault derivative of the projection onto the cone of positive semidefinite matrices in $\mathcal{S}^n$ (we use $\mathcal{S}^n$ instead of $\mathcal{S}^p$, to avoid double usage of p). For the construction of the Thibault derivative, we make use of the spectral decomposition of a matrix and a certain set $\mathcal{M}_H$, which is defined below.

Definition 3.24. Let $H, X \in \mathcal{S}^n$ and $\operatorname{diag}\Lambda$ is the vector with eigenvalues λ_i of X arranged in nonincreasing order. I_p, I_m, J are index sets of these positive, negative, and zero eigenvalues, respectively, $N := \{1, \ldots, n\}$, $z := \#J$, $m := \#I_m$, and $p := \#I_p$.
We define $\mathcal{M}_H$ as the set of matrices $M \in \mathcal{S}^z$ such that we have $(P_M, \Lambda) \in \mathcal{SPEC}(X)$, $\tilde{H} = P_M H P_M^T$ and

$$M = (\alpha_{ij})_{JJ} \circ (\tilde{H}^2)_{JJ} + \begin{pmatrix} \tilde{H}_{\tilde{I}_{(+)}\tilde{I}_{(+)}} & \tilde{H}_{\tilde{I}_{(+)}\tilde{J}} & (\beta_{ij})_{\tilde{I}_{(+)}\tilde{I}_{(-)}} \circ \tilde{H}_{\tilde{I}_{(+)}\tilde{I}_{(-)}} \\ \tilde{H}_{\tilde{J}\tilde{I}_{(+)}} & (\star\star) & 0 \\ ((\beta_{ij})_{\tilde{I}_{(+)}\tilde{I}_{(-)}} \circ \tilde{H}_{\tilde{I}_{(+)}\tilde{I}_{(-)}})^T & 0 & 0 \end{pmatrix}$$

with $\alpha_{ij} = \alpha_{ji} \in [0, +\infty)$, $\beta_{ij} \in (0, 1)$. $\tilde{I}_{(+)}$, $\tilde{I}_{(-)}$, and $\tilde{J}$ are any index sets that fulfill $\tilde{I}_{(+)} \cup \tilde{I}_{(-)} \cup \tilde{J} = \{p+1, \ldots, p+z\}$, $\tilde{z} := \#\tilde{J}$, $\bar{I} := N \setminus \tilde{J}$ and

$$(\star\star) = \frac{1}{2}\left(\left[\tilde{H}^2_{\tilde{J}\tilde{J}} - \left(\tilde{H}_{\tilde{J}\bar{I}} + (\tilde{H}^2)_{\tilde{J}\bar{I}}A\right)^2 + \tilde{H}^2_{\tilde{J}\bar{I}}\right]^{\frac{1}{2}} + \tilde{H}_{\tilde{J}\tilde{J}}\right) - (\alpha_{ij})_{\tilde{J}\tilde{J}} \circ (\tilde{H}^2)_{\tilde{J}\tilde{J}},$$

where $A = (a_{ij})_{\bar{I}\bar{I}}$ is a diagonal matrix with $a_{ii} \in \mathbb{R}$ for $i \in \tilde{I}_{(+)} \cup \tilde{I}_{(-)}$ and else zeros.

Theorem 3.25 gives us a representation for the elements of the Thibault derivative of the projection onto the cone of positive semidefinite matrices in $\mathcal{S}^n$.

Theorem 3.25. *Let $\Pi_+ : \mathcal{S}^n \to \mathcal{S}^n$ be defined by $\Pi_+(X) = X_+$. Then, each element of the set $T\Pi_+(X)(H)$ of the Thibault derivative of Π_+ at any $X \in \mathcal{S}^n$ and for any $H \in \mathcal{S}^n$ belongs to the set of matrices*

$$P_M^T \begin{pmatrix} \tilde{H}_{I_pI_p} & \tilde{H}_{I_pJ} & \left(\frac{\lambda_i}{\lambda_i + |\lambda_j|}\right)_{I_pI_m} \circ \tilde{H}_{I_pI_m} \\ (\tilde{H}_{I_pJ})^T & M & 0 \\ \left(\left(\frac{\lambda_i}{\lambda_i + |\lambda_j|}\right)_{I_pI_m} \circ \tilde{H}_{I_pI_m}\right)^T & 0 & 0 \end{pmatrix} P_M$$

with $M \in \mathcal{M}_H$.

Proof. Consider $X^k \in \mathcal{S}^n$ near X, such that $X^k \to X$, and p (m) the number of positive (negative) eigenvalues of X. Let us choose $(Q^k, \Gamma^k) \in \mathcal{SPEC}(X^k)$ in order to define

$$Q^k X^k (Q^k)^T = \Gamma^k = \begin{pmatrix} \Gamma_p^k & & & & \\ & \Gamma_m^k & & & \\ & & \Gamma_{\varepsilon+}^k & & \\ & & & \Gamma_{\varepsilon-}^k & \\ & & & & \Gamma_{\varepsilon 0}^k \end{pmatrix}, \tag{3.9}$$

for X^k sufficiently close to X. The diagonal matrix Γ^k is constructed as described below. The vector $\gamma^k := \operatorname{diag} \Gamma^k$ is the vector of eigenvalues γ_i^k, $\forall i \in \{1, \ldots, n\}$, of X^k, however, in this case the eigenvalues are not sorted throughout.
To construct $\Gamma_{\varepsilon+}^k, \Gamma_{\varepsilon-}^k$, and $\Gamma_{\varepsilon 0}^k$ fix $\varepsilon > 0$ that fulfills

$$\min_{\lambda_i \neq 0} |\lambda_i| > 2\varepsilon,$$

where λ_i are the eigenvalues of X. $\Gamma_{\varepsilon+}^k$ $(\Gamma_{\varepsilon-}^k)$ is the diagonal matrix with all positive (negative) eigenvalues of X^k smaller (larger) than ε $(-\varepsilon$, respectively) and $\Gamma_{\varepsilon 0}^k$ is the remaining matrix with the zero eigenvalues. Consequently, Γ_p^k and Γ_m^k are diagonal matrices with eigenvalues not smaller than ε and not larger than $-\varepsilon$, respectively.
For sufficiently large k, Γ^k, Γ_p^k, and Γ_m^k stay constant in size. The matrices $\Gamma_{\varepsilon+}^k, \Gamma_{\varepsilon-}^k$, and $\Gamma_{\varepsilon 0}^k$ may change in size, however, the sum of the number of their rows (or columns) always adds up to z. We choose $(Q, \Gamma) \in \mathcal{SPEC}(X)$ such that $\lim_{k\to\infty} \Gamma^k = \Gamma$ as follows.
By taking a subsequence, if necessary, we can assume that $\{Q^k\}$ is a convergent sequence with the limit $\bar{Q} \equiv \lim_{k\to\infty} Q^k$. Then

$$X = \lim_{k\to\infty} X^k = \lim_{k\to\infty} (Q^k)^T \Gamma^k Q^k = \bar{Q}^T \Gamma \bar{Q},$$

and we can identify $\bar{Q}$ with Q. Note, that Q is dependent on the order of the eigenvalues in $\operatorname{diag} \Gamma^k$ for sufficiently large k. For $z > 1$, there are several possible matrices for Q.
If X^k is sufficiently close to $X = Q^T \Gamma Q$, then the matrix Γ_p^k is a $p \times p$ matrix and Γ_m^k an $m \times m$ matrix, respectively. This justifies the definition in (3.9)

for X^k near X. Furthermore, we denote

$$D^k := \begin{pmatrix} \Gamma_p^k & & \\ & \Gamma_m^k & \\ & & \Gamma_{\varepsilon\pm}^k \end{pmatrix},$$

which is regular, and $\Gamma_{\varepsilon\pm}^k$ is the diagonal matrix constructed with $\Gamma_{\varepsilon+}^k$ and $\Gamma_{\varepsilon-}^k$. We define

$$C^k := \left|D^k\right|.$$

The body of the proof consists of three parts. In the first part, we reformulate the difference $\Pi_+(X^k + t_k H) - \Pi_+(X^k)$ in definition 3.17 of the Thibault derivative to a suitable term. In the second part, we study the block matrices of this difference and its convergence characteristics. In the final part, we show what happens, when this reformulation converges.

Part 1

Assume $H \neq 0$, otherwise Theorem 3.25 follows trivially. Let

$$\begin{aligned} \Delta(k) &:= (X^k + t_k H)_+ - (X^k)_+ \\ &= \frac{1}{2}\left(X^k + t_k H + \left|X^k + t_k H\right| - X^k - \left|X^k\right|\right), \end{aligned}$$

then, since $Q^k \left|X^k\right| (Q^k)^T = \left|\Gamma^k\right|$ and

$$Q^k \left|X^k + t_k H\right| (Q^k)^T = \left|Q^k X^k (Q^k)^T + t_k Q^k H (Q^k)^T\right|$$

by Lemma 2.4 (ix) we get

$$2\tilde{\Delta}(k) := 2Q^k \Delta(k)(Q^k)^T = \left|\Gamma^k + t_k \tilde{H}^k\right| - \left|\Gamma^k\right| + t_k \tilde{H}^k, \tag{3.10}$$

where $\tilde{H}^k := Q^k H (Q^k)^T$.

Let $I^k = I_{(+)}^k \cup I_{(-)}^k$ be the union of the index sets of positive and negative eigenvalues of X^k, and let J^k be the set of its zero eigenvalues, respectively. We have $N = I^k \cup J^k$ for any k.

Let us concentrate on

$$z(k) := \left|\Gamma^k + t_k \tilde{H}^k\right| - \left|\Gamma^k\right| = \sqrt{(\Gamma^k + t_k \tilde{H}^k)^2} - \sqrt{(\Gamma^k)^2}. \tag{3.11}$$

We have $\big(\Gamma^k + t_k\tilde{H}^k\big)^2 = \big(\left|\Gamma^k\right|\big)^2 + W^k \succeq 0$ where $\left|\Gamma^k\right|$ is the $n \times n$ matrix with C^k in the top left corner and else zeros, and W^k is defined as follows (for convenience we write I, J instead of I^k, J^k)

$$\begin{aligned}
W^k =& \begin{pmatrix} W^k_{II} & W^k_{IJ} \\ W^k_{JI} & W^k_{JJ} \end{pmatrix} \\
=& t_k(\Gamma^k\tilde{H}^k + \tilde{H}^k\Gamma^k) + t_k^2(\tilde{H}^k)^2 \\
=& t_k \begin{pmatrix} D^k\tilde{H}^k_{II} + \tilde{H}^k_{II}D^k & D^k\tilde{H}^k_{IJ} \\ (D^k\tilde{H}^k_{IJ})^T & 0 \end{pmatrix} \\
&+ t_k^2 \begin{pmatrix} (\tilde{H}^k_{II})^2 + \tilde{H}^k_{IJ}\tilde{H}^k_{JI} & \tilde{H}^k_{II}\tilde{H}^k_{IJ} + \tilde{H}^k_{IJ}\tilde{H}^k_{JJ} \\ (\tilde{H}^k_{II}\tilde{H}^k_{IJ} + \tilde{H}^k_{IJ}\tilde{H}^k_{JJ})^T & \tilde{H}^k_{JI}\tilde{H}^k_{IJ} + (\tilde{H}^k_{JJ})^2 \end{pmatrix}.
\end{aligned} \tag{3.12}$$

Note, that $z(k) = \sqrt{(|\Gamma^k|)^2 + W^k} - \left|\Gamma^k\right|$ which is equivalent to $\big(z(k) + \left|\Gamma^k\right|\big)^2 = \big(\left|\Gamma^k\right|\big)^2 + W^k$, and expanded we get

$$z(k)\left|\Gamma^k\right| + \left|\Gamma^k\right| z(k) = W^k - z(k)z(k)^T. \tag{3.13}$$

Part 2
Consider $\tilde{\Delta}(k)$ as a block matrix for some fixed k sufficiently large. We look at its matrices $\tilde{\Delta}_{I^kI^k}$, $\tilde{\Delta}_{I^kJ^k}$ and $\tilde{\Delta}_{J^kJ^k}$.

Block matrix $\tilde{\Delta}_{I^kI^k}$
From Lemma 2.6 [52, Lemma 6.2]) and (3.12), it follows that

$$\begin{aligned}
z(k)_{I^kI^k} &= L^{-1}_{C^k}(W^k_{I^kI^k}) + o\left(\left\|W^k\right\|\right) \\
&= t_kL^{-1}_{C^k}\big(D^k\tilde{H}^k_{I^kI^k} + \tilde{H}^k_{I^kI^k}D^k + t_k(\tilde{H}^k)^2_{I^kI^k}\big) + o\left(\left\|W^k\right\|\right),
\end{aligned}$$

where $L^{-1}_{C^k}$ is the inverse operator of the Lyapunov operator $L_{C^k}(Y) = C^kY + YC^k$, for any symmetric matrix Y of appropriate size. $L^{-1}_{C^k}$ is invertible because C^k is positive definite. This leads to

$$\begin{aligned}
&t_kL^{-1}_{C^k}\left(D^k\tilde{H}^k_{I^kI^k} + \tilde{H}^k_{I^kI^k}D^k + t_k(\tilde{H}^k)^2_{I^kI^k}\right) \\
&\qquad = 2\tilde{\Delta}(k)_{I^kI^k} - t_k\tilde{H}^k_{I^kI^k} + o\left(\left\|W^k\right\|\right),
\end{aligned}$$

which is equivalent to

$$
\begin{aligned}
t_k\left(D^k\tilde{H}^k_{I^kI^k}+\tilde{H}^k_{I^kI^k}D^k\right) = & \, 2C^k\tilde{\Delta}(k)_{I^kI^k}+2\tilde{\Delta}(k)_{I^kI^k}C^k - t_kC^k\tilde{H}^k_{I^kI^k} \\
& - t_k\tilde{H}^k_{I^kI^k}C^k + C^k o\left(\left\|W^k\right\|\right) + o\left(\left\|W^k\right\|\right)C^k \\
& - t_k^2(\tilde{H}^k)^2_{I^kI^k}.
\end{aligned}
$$

For any $i,j\in I^k$, this means

$$
\begin{aligned}
& \frac{2}{t_k}\left(\left|\gamma_i^k\right|\tilde{\Delta}(k)_{ij}+\tilde{\Delta}(k)_{ij}\left|\gamma_j^k\right|\right) \\
& = \gamma_i^k\tilde{h}^k_{ij}+\tilde{h}^k_{ij}\gamma_j^k+\left|\gamma_i^k\right|\tilde{h}^k_{ij}+\tilde{h}^k_{ij}\left|\gamma_j^k\right|+\left(\left|\gamma_i^k\right|+\left|\gamma_j^k\right|\right)\left(\frac{o(\left\|W^k\right\|)}{t_k}\right)_{ij} \\
& \quad + t_k(\tilde{H}^k)^2_{ij},
\end{aligned}
$$

and we get

$$
\begin{aligned}
\frac{\tilde{\Delta}(k)_{ij}}{t_k} = & \frac{1}{2}\frac{\gamma_i^k+\left|\gamma_i^k\right|+\gamma_j^k+\left|\gamma_j^k\right|}{\left|\gamma_i^k\right|+\left|\gamma_j^k\right|}\tilde{h}^k_{ij}+\frac{1}{2}\frac{t_k(\tilde{H}^k)^2_{ij}}{\left|\gamma_i^k\right|+\left|\gamma_j^k\right|}+\left(\frac{o(\left\|W^k\right\|)}{t_k}\right)_{ij} \\
= & \frac{\max\{0,\gamma_i^k\}+\max\{0,\gamma_j^k\}}{\left|\gamma_i^k\right|+\left|\gamma_j^k\right|}\tilde{h}^k_{ij} \\
& + \frac{1}{2}\frac{t_k(\tilde{H}^k)^2_{ij}}{\left|\gamma_i^k\right|+\left|\gamma_j^k\right|}+\left(\frac{o(\left\|W^k\right\|)}{t_k}\right)_{ij}.
\end{aligned}
\tag{3.14}
$$

Note, that for $t_k\to 0$ and $k\to\infty$ we have $\left\|W^k\right\|\to 0$ and $\frac{o(\left\|W^k\right\|)}{t_k}\to 0$. Either (case 1) $(\left|\gamma_i^k\right|+\left|\gamma_j^k\right|)>\varepsilon$ for all k, hence, at least one of the eigenvalues converges towards a nonzero eigenvalue, or (case 2) $(\left|\gamma_i^k\right|+\left|\gamma_j^k\right|)\to 0$ when both eigenvalues converge towards zero. In the first case, we have

$$
\lim_{k\to\infty}\frac{\tilde{\Delta}(k)_{ij}}{t_k} = \left(\lim_{k\to\infty}\frac{\max\{0,\gamma_i^k\}+\max\{0,\gamma_j^k\}}{\left|\gamma_i^k\right|+\left|\gamma_j^k\right|}\right)\tilde{h}_{ij}.
$$

Assuming that both eigenvalues eventually converge towards zero (case 2), we must take a closer look at

$$
\zeta^k_{ij} := \frac{t_k}{\left|\gamma_i^k\right|+\left|\gamma_j^k\right|}.
$$

If $(|\gamma_i^k| + |\gamma_j^k|)$ converges faster than t_k, then $\lim_{k\to\infty} \frac{1}{t_k}\tilde{\Delta}(k)_{ij} = \infty$, and we are not looking at an element in the Thibault set. Hence, we assume that t_k converges at least as fast as $|\gamma_i^k|+|\gamma_j^k|$. Then, ζ_{ij}^k can converge to any $\zeta_{ij} \geq 0$, depending on the choice of t_k, and we have

$$\lim_{k\to\infty} \frac{\tilde{\Delta}(k)_{ij}}{t_k} = \left(\lim_{k\to\infty} \frac{\max\{0,\gamma_i^k\} + \max\{0,\gamma_j^k\}}{|\gamma_i^k| + |\gamma_j^k|}\right) \tilde{h}_{ij} + \frac{\zeta_{ij}}{2}(\tilde{H}^2)_{ij}.$$

Block matrix $\tilde{\Delta}_{I^kJ^k}$

To determine $\tilde{\Delta}(k)_{I^kJ^k}$, we take a similar approach as above for $\tilde{\Delta}(k)_{I^kI^k}$. Let us choose k big enough such that $J^k \subseteq J$. From (3.13), we get for any i and j

$$\left(|\gamma_i^k| + |\gamma_j^k|\right) z(k)_{ij} = W_{ij}^k - \left(z(k)z(k)^T\right)_{ij}. \tag{3.15}$$

For $i \in I^k$ and $j \in J^k$, we have from (3.12) $W_{ij}^k = t_k\gamma_i^k\tilde{H}_{ij}^k + t_k^2((\tilde{H}^k)^2)_{ij}$, $\gamma_j^k = 0$, and when applied to (3.15), we get

$$z(k)_{ij} = \frac{1}{|\gamma_i^k|}\left(t_k\gamma_i^k\tilde{H}_{ij}^k + t_k^2((\tilde{H}^k)^2)_{ij} - (z(k)z(k)^T)_{ij}\right).$$

Furthermore, by using (3.11) and $N = I^k \cup J^k$, we get

$$\begin{aligned}
z(k)z(k)^T =& \left(\left|\Gamma^k + t_k\tilde{H}^k\right| - |\Gamma^k|\right)^2 \\
=& t_k^2(\tilde{H}^k)^2 + (\Gamma^k + t_k\tilde{H}^k)\Gamma^k + \Gamma^k(\Gamma^k + t_k\tilde{H}^k) \\
&- \left|\Gamma^k + t_k\tilde{H}^k\right| |\Gamma^k| - |\Gamma^k| \left|\Gamma^k + t_k\tilde{H}^k\right| \\
=& t_k^2(\tilde{H}^k)^2 + (\gamma_i^k + \gamma_j^k)_{NN} \circ (\Gamma^k + t_k\tilde{H}^k) \\
&- \left(|\gamma_i^k| + |\gamma_j^k|\right)_{NN} \circ \left|\Gamma^k + t_k\tilde{H}^k\right|,
\end{aligned}$$

and, hence (IJ is I^kJ^k), with Lemma 2.4 we get

$$\begin{aligned}
z(k)_{IJ} =& t_k(C^k)^{-1}D^k\tilde{H}_{IJ}^k + t_k^2(C^k)^{-1}((\tilde{H}^k)^2)_{IJ} \\
&- (C^k)^{-1}\Big[(t_k^2(\tilde{H}^k)^2)_{IJ} + D^k(\Gamma^k + t_k\tilde{H}^k)_{IJ} \\
&- C^k\left|\Gamma^k + t_k\tilde{H}^k\right|_{IJ}\Big].
\end{aligned}$$

From Lemma 2.6 by Tseng [52, Lemma 6.2], one has by (3.12)

$$\begin{aligned} z(k)_{I^kJ^k} =&(C^k)^{-1}W^k_{I^kJ^k} + o(\|W^k\|) \\ =&t_k(C^k)^{-1}D^k\tilde{H}^k_{I^kJ^k} + t_k^2(C^k)^{-1}((\tilde{H}^k)^2)_{I^kJ^k} + o(\|W^k\|), \end{aligned}$$

hence,

$$\begin{aligned} &(C^k)^{-1}\left[(t_k^2(\tilde{H}^k)^2)_{IJ} + D^k(\Gamma^k + t_k\tilde{H}^k)_{IJ} - C^k\left|\Gamma^k + t_k\tilde{H}^k\right|_{IJ}\right] \\ &= o(\|W^k\|). \end{aligned}$$

With $2\tilde{\Delta}(k) = z(k) + t_k\tilde{H}^k$, we then write

$$\begin{aligned} 2\tilde{\Delta}(k)_{I^kJ^k} =&t_k(C^k)^{-1}D^k\tilde{H}^k_{I^kJ^k} + t_k\tilde{H}^k_{I^kJ^k} \\ &+ t_k^2(C^k)^{-1}(\tilde{H}^k)^2_{I^kJ^k} + o(\|W^k\|). \end{aligned} \tag{3.16}$$

Define by

$$E^k := \left((C^k)^{-1}D^k\right)_{ij} = \begin{cases} 1 & i=j, \gamma_i^k > 0 \\ -1 & i=j, \gamma_i^k < 0 \\ 0 & i \neq j. \end{cases}$$

This is a diagonal matrix with 1 and -1 in the diagonal and, depending on γ_i^k, we get $(E^k\tilde{H}^k_{I^kJ^k} + \tilde{H}^k_{I^kJ^k})_{ij} = 2\tilde{h}^k_{ij}$ or 0. From (3.16), we get

$$\frac{1}{t_k}\tilde{\Delta}(k)_{ij} = \begin{cases} \tilde{h}^k_{ij} + \frac{t_k}{2|\gamma_i^k|}(\tilde{H}^k)^2_{ij} + \left(\frac{o(\|W^k\|)}{t_k}\right)_{ij} & \gamma_i^k > 0 \\ \frac{t_k}{2|\gamma_i^k|}(\tilde{H}^k)^2_{ij} + \left(\frac{o(\|W^k\|)}{t_k}\right)_{ij} & \gamma_i^k < 0. \end{cases} \tag{3.17}$$

We must distinguish between different cases, depending on how fast the rate of convergence is for C^k when it contains eigenvalues that converge towards zero.
If $\left|\gamma_i^k\right|$ converges faster than t_k, then $\lim_{k\to\infty}\frac{1}{t_k}\tilde{\Delta}(k)_{ij} = \infty$, and we are not looking at an element in the Thibault set. Hence, if $\left|\gamma_i^k\right| \to 0$ then t_k must converge at least as fast as $\left|\gamma_i^k\right|$, and (since $\gamma_j^k = 0$) we have

$$\lim_{k\to\infty}\frac{t_k}{2\left|\gamma_i^k\right|} = \lim_{k\to\infty}\frac{\zeta^k_{ij}}{2} = \frac{\zeta_{ij}}{2},$$

where $\zeta_{ij} \geq 0$ depends on the choice of t_k. Then, for sufficiently large k, we get

$$\lim_{k\to\infty} \frac{1}{t_k}\tilde{\Delta}(k)_{ij} = \begin{cases} \tilde{h}_{ij} + \frac{\zeta_{ij}}{2}\tilde{H}_{ij}^2 & \gamma_i^k > 0 \\ \frac{\zeta_{ij}}{2}\tilde{H}_{ij}^2 & \gamma_i^k < 0. \end{cases}$$

Block matrix $\tilde{\Delta}_{J^k J^k}$
For the construction of $\tilde{\Delta}(k)_{J^k J^k}$ for sufficiently large k, we have by $\gamma_i^k = \gamma_j^k = 0$ and (3.13)

$$W^k_{J^k J^k} = \left(z(k)z(k)^T\right)_{J^k J^k} = z(k)^T_{I^k J^k} z(k)_{I^k J^k} + z(k)^2_{J^k J^k}.$$

From the formulation of $z(k)_{I^k J^k}$ in (3.16), this implies (IJ is $I^k J^k$)

$$\begin{aligned} z(k)^2_{JJ} = & W^k_{JJ} - z(k)^T_{IJ} z(k)_{IJ} \\ =& t_k^2\left((\tilde{H}^k_{JI})^2 + (\tilde{H}^k_{JJ})^2\right) \\ & - \left(t_k E^k \tilde{H}^k_{IJ} + t_k^2 (C^k)^{-1}((\tilde{H}^k)^2)_{IJ} + o(\|W^k\|)\right)^T \\ & \left(t_k E^k \tilde{H}^k_{IJ} + t_k^2 (C^k)^{-1}((\tilde{H}^k)^2)_{IJ} + o(\|W^k\|)\right) \\ =& t_k^2(\tilde{H}^k_{JI})^2 + t_k^2(\tilde{H}^k_{JJ})^2 - t_k^2(\tilde{H}^k_{JI})^2 \\ & - t_k^3(\tilde{H}^k_{IJ})^T(D^k)^{-1}((\tilde{H}^k)^2)_{IJ} - t_k^3((\tilde{H}^k)^2)^T_{IJ}(D^k)^{-1}\tilde{H}^k_{IJ} \\ & - t_k^4\left(((\tilde{H}^k)^2)_{JI}(C^k)^{-1}\right)^2 \\ & - t_k(\tilde{H}^k_{IJ})^T E^k o(\|W^k\|) - o(\|W^k\|) t_k E^k \tilde{H}^k_{IJ} \\ & - t_k^2((\tilde{H}^k)^2)^T_{IJ}(C^k)^{-1} o(\|W^k\|) - o(\|W^k\|) t_k^2 (C^k)^{-1}((\tilde{H}^k)^2)_{IJ} \\ & - o(\|W^k\|^2). \end{aligned} \tag{3.18}$$

Consider $A^k := t_k(D^k)^{-1}$ i.e. A^k is a diagonal matrix with diagonals $\frac{t_k}{\gamma_i^k}$, then

$$\begin{aligned}
z(k)^2_{JJ} = t_k^2\Big[&(\tilde{H}^k_{JJ})^2 - (\tilde{H}^k_{IJ})^T A^k((\tilde{H}^k)^2)_{IJ} - ((\tilde{H}^k)^2)^T_{IJ} A^k \tilde{H}^k_{IJ} \\
&- ((\tilde{H}^k)^2)^2_{JI}(A^k)^2 \\
&- (\tilde{H}^k_{IJ})^T E^k \frac{o(\|W^k\|)}{t_k} - \frac{o(\|W^k\|)}{t_k} E^k \tilde{H}^k_{IJ} \\
&- ((\tilde{H}^k)^2)^T_{IJ} \left|A^k\right| \frac{o(\|W^k\|)}{t_k} - \frac{o(\|W^k\|)}{t_k} \left|A^k\right| ((\tilde{H}^k)^2)_{IJ} \\
&- \frac{o(\|W^k\|^2)}{t_k^2}\Big] \\
= t_k^2\Big[&(\tilde{H}^k_{JJ})^2 - (\tilde{H}^k_{IJ})^T A^k((\tilde{H}^k)^2)_{IJ} - ((\tilde{H}^k)^2)^T_{IJ} A^k \tilde{H}^k_{IJ} \\
&- ((\tilde{H}^k)^2)^2_{JI}(A^k)^2 + \frac{o(\|W^k\|)}{t_k}\Big]
\end{aligned}$$

and

$$\begin{aligned}
&\frac{1}{t_k}\tilde{\Delta}(k)_{JJ} = \frac{1}{2}\left(\frac{1}{t_k} z(k)_{JJ} + \tilde{H}^k_{JJ}\right) \\
&= \frac{1}{2}\left(\left[(\tilde{H}^k_{JJ})^2 - \left(\tilde{H}^k_{JI} + ((\tilde{H}^k)^2)_{JI} A^k\right)^2 + (\tilde{H}^k_{JI})^2 - \frac{o(\|W^k\|)}{t_k}\right]^{\frac{1}{2}}\right. \qquad (3.19) \\
&\left. + \tilde{H}^k_{JJ}\right).
\end{aligned}$$

For a sufficiently large k (then $I^k = I^{k+1}$), we define $\bar{I} := I^k$, $\tilde{J} := J^k$ and $A := A_{\bar{I}\bar{I}}$ such that $\lim_{k\to\infty}(A^k)_{I^k I^k} = A$.
That is

$$\lim_{k\to\infty} \frac{\tilde{\Delta}(k)_{J^k J^k}}{t_k} = \frac{1}{2}\left(\left[\tilde{H}^2_{\tilde{J}\tilde{J}} - \left(\tilde{H}_{\tilde{J}\bar{I}} + (\tilde{H}^2)_{\tilde{J}\bar{I}} A\right)^2 + \tilde{H}^2_{\tilde{J}\bar{I}}\right]^{\frac{1}{2}} + \tilde{H}_{\tilde{J}\tilde{J}}\right).$$

Note, that A must be bounded.
If $\lim \frac{t_k}{\gamma_i^k} \to \infty$ for $k \to \infty$, then we are not looking at an element in the Thibault set.

Part 3

Assuming $\lim_{k\to\infty} \frac{1}{t_k}\Delta(k)$ exists, we give a formula for its value.
We need a different spectral decomposition of X and X^k, where $X^k \to X$. Let $(P,\Lambda) \in \mathcal{SPEC}(X)$ and $(P^k,\Lambda^k) \in \mathcal{SPEC}(X^k)$ such that the eigenvalues in Λ are sorted by size in descending order and $\lim_{k\to\infty} \Lambda^k = \Lambda$. Of course, Λ is just the matrix with a permutation of the diagonal of Γ. We see that P is dependent on the order of the eigenvalues in $\operatorname{diag}\Lambda^k$ for sufficiently large k. For $z > 1$ there are several possible matrices for P. Let k be sufficiently large, then we have

$$\Lambda^k = \begin{pmatrix} \Lambda_p^k & & \\ & \Lambda_\varepsilon^k & \\ & & \Lambda_m^k \end{pmatrix},$$

where the eigenvalues of Λ_p^k (Λ_m^k) are not smaller (not bigger) than ε ($-\varepsilon$), and the eigenvalues of Λ_ε^k are in absolute value smaller than ε. Let $\Lambda_p \in \mathcal{S}^p$, $\Lambda_m \in \mathcal{S}^m$, and $\Lambda_z \in \mathcal{S}^z$ be the diagonal matrices that only contain the positive, negative and zero, respectively, eigenvalues of X. We easily see that $\operatorname{rank}\Lambda_p^k = \operatorname{rank}\Lambda_p$ and $\operatorname{rank}\Lambda_m^k = \operatorname{rank}\Lambda_m$, respectively. Note, that the eigenvalues of Λ^k are not necessarily sorted by size and for the re-sorted eigenvalues, we have the index sets $\hat{I}_{(+)}^k$, $\hat{I}_{(-)}^k$, and $\hat{J}^k$. Let us assume w.l.o.g. that

$$\Lambda_\varepsilon^k = \begin{pmatrix} \Lambda_{\varepsilon+}^k & & \\ & \Lambda_{\varepsilon 0}^k & \\ & & \Lambda_{\varepsilon-}^k \end{pmatrix}.$$

By taking a subsequence, if necessary, we assume that $\{P^k\}$ is a convergent sequence with the limit $\bar{P} \equiv \lim_{k\to\infty} P^k$. Then,

$$X = \lim_{k\to\infty} X^k = \lim_{k\to\infty} (P^k)^T \Lambda^k P^k = (P)^T \Lambda P,$$

and we can identify $\bar{P}$ with P. Let now $\tilde{H}^k = P^k H (P^k)^T$[2], then, from (3.14) we get

$$\frac{1}{t_k}\tilde{\Delta}(k)_{ij} = \tilde{h}_{ij} + \frac{t_k}{2}\frac{(\tilde{H}^k)_{ij}^2}{|\lambda_i^k| + |\lambda_j^k|} + \left(\frac{o(\|W^k\|)}{t_k}\right)_{ij}$$

[2] W.l.o.g. the terms W^k, D^k, C^k and A^k from part 1 and 2 are considered with this new order.

if $i, j \in \hat{I}^k_{(+)}$,

$$\frac{1}{t_k}\tilde{\Delta}(k)_{ij} = \frac{\lambda_i^k}{\left|\lambda_i^k\right| + \left|\lambda_j^k\right|}\tilde{h}_{ij} + \frac{t_k}{2}\frac{(\tilde{H}^k)^2_{ij}}{\left|\lambda_i^k\right| + \left|\lambda_j^k\right|} + \left(\frac{o(\left\|W^k\right\|)}{t_k}\right)_{ij}$$

if $i \in \hat{I}^k_{(+)}$ and $j \in \hat{I}^k_{(-)}$ and finally

$$\frac{1}{t_k}\tilde{\Delta}(k)_{ij} = 0 + \frac{t_k}{2}\frac{(\tilde{H}^k)^2_{ij}}{\left|\lambda_i^k\right| + \left|\lambda_j^k\right|} + \left(\frac{o(\left\|W^k\right\|)}{t_k}\right)_{ij}$$

if $i, j \in \hat{I}^k_{(-)}$.
Together with (3.17) and (3.19), we get the following representation

$$\begin{aligned}
\frac{1}{t_k}\Delta(k) = {}& (P^k)^T \\
&\left[\begin{pmatrix} \tilde{H}^k_{(++)} & \tilde{H}^k_{(+0)} & \left(\frac{\lambda_i^k}{\lambda_i^k + \left|\lambda_j^k\right|}\right)_{(\pm)} \circ \tilde{H}^k_{(\pm)} \\ (\tilde{H}^k_{(+0)})^T & \frac{1}{2}\left(\sqrt{(\tilde{H}^k_{(00)})^2 - \star} + \tilde{H}^k_{(00)}\right) & 0 \\ \left(\left(\frac{\lambda_i^k}{\lambda_i^k + \left|\lambda_j^k\right|}\right)_{(\pm)} \circ \tilde{H}^k_{(\pm)}\right)^T & 0 & 0 \end{pmatrix}\right. \\
&+ \frac{t_k}{2}\begin{pmatrix} \left(\frac{(\tilde{H}^k)^2}{\lambda_i^k + \lambda_j^k}\right)_{(++)} & \left(\frac{(\tilde{H}^k)^2}{\lambda_i^k}\right)_{(+0)} & \left(\frac{(\tilde{H}^k)^2}{\lambda_i^k + \left|\lambda_j^k\right|}\right)_{(\pm)} \\ \left(\frac{(\tilde{H}^k)^2}{\lambda_j^k}\right)_{(0+)} & 0 & \left(\frac{(\tilde{H}^k)^2}{\left|\lambda_j^k\right|}\right)_{(0-)} \\ \left(\frac{(\tilde{H}^k)^2}{\left|\lambda_i^k\right| + \lambda_j^k}\right)_{(\mp)} & \left(\frac{(\tilde{H}^k)^2}{\left|\lambda_i^k\right|}\right)_{(-0)} & \left(\frac{(\tilde{H}^k)^2}{\left|\lambda_i^k\right| + \left|\lambda_j^k\right|}\right)_{(--)} \end{pmatrix} \\
&\left. + \frac{1}{t_k}\begin{pmatrix} o(\left\|W^k\right\|) & o(\left\|W^k\right\|) & o(\left\|W^k\right\|) \\ o(\left\|W^k\right\|)^T & 0 & o(\left\|W^k\right\|) \\ o(\left\|W^k\right\|)^T & o(\left\|W^k\right\|)^T & o(\left\|W^k\right\|) \end{pmatrix}\right] P^k,
\end{aligned} \tag{3.20}$$

where we have the abbreviations $(++) := \hat{I}^k_{(+)}\hat{I}^k_{(+)}$, $(--) := \hat{I}^k_{(-)}\hat{I}^k_{(-)}$, $(+0) := \hat{I}^k_{(+)}\hat{J}^k$, $(-0) := \hat{I}^k_{(-)}\hat{J}^k$, $(\pm) := \hat{I}^k_{(+)}\hat{I}^k_{(-)}$, $(00) := \hat{J}^k\hat{J}^k$ and

$$(\star) := \left(\tilde{H}^k_{JI} + ((\tilde{H}^k)^2)_{JI}A^k\right)^2 - (\tilde{H}^k_{JI})^2 + \frac{o(\left\|W^k\right\|)}{t_k}$$

with $IJ = \hat{I}^k \hat{J}^k$. Note, that $(\star)$ can be smaller than a $z \times z$ matrix.
As mentioned above, all eigenvalues of Λ_p^k (Λ_m^k) stay positive (negative, respectively) for X^k sufficiently close to X.
Assume that $X^k \to X$ and for X^k the eigenvalues λ_i of Λ_ε^k for certain i change their sign when they converge to 0, hence,

$$\forall m \in \mathbb{N} \quad \exists s, t > m : \lambda_i(X^s) > 0, \lambda_i(X^t) < 0.$$

Then $\lim_{k\to\infty} \frac{\Delta(k)}{t_k}$ does not exist for chosen X^k. This is easily proven (cf. Malick and Sendov [36, Lemma 2.12(b)] for a proof).
Hence, if we assume that $\lim_{k\to\infty} \frac{1}{t_k}\Delta(k)$ exists, then it has the form

$$P_M^T \begin{pmatrix} \tilde{H}_{I_pI_p} & \tilde{H}_{I_pJ} & \left(\frac{\lambda_i}{\lambda_i+|\lambda_j|}\right)_{I_pI_m} \circ \tilde{H}_{I_pI_m} \\ (\tilde{H}_{I_pJ})^T & M & 0 \\ \left(\left(\frac{\lambda_i}{\lambda_i+|\lambda_j|}\right)_{I_pI_m} \circ \tilde{H}_{I_pI_m}\right)^T & 0 & 0 \end{pmatrix} P_M$$

with a matrix $M \in \mathcal{S}^z$, and P_M depends on the order of the eigenvalues converging towards zero. The index sets $\hat{I}^k_{(+)}$ ($\hat{I}^k_{(-)}$) can be bigger than I_p (I_m) for all k.
To get the structure of M, we have to take a closer look at the behavior of the sequence X^k in relation to the eigenvalues that converge towards zero.
If all eigenvalues of Λ_ε^k are equal zero for X^k near X, then $M = (\tilde{H}_{JJ})_+$. Suppose, these eigenvalues are also greater and/or smaller than zero, then for k sufficiently large we proceed with a construction based on the second summand of (3.20). Similar to above, let

$$\tilde{\zeta}_{ij}^k := \begin{cases} \frac{t_k}{|\lambda_i^k|+|\lambda_j^k|} & \text{if } |\lambda_i^k| + |\lambda_j^k| \neq 0, \\ 0 & \text{else,} \end{cases}$$

then M_{ij} contains $\lim \frac{1}{2}\tilde{\zeta}_{ij}^k(\tilde{H}^k)^2_{ij} = \frac{1}{2}\tilde{\zeta}_{ij}\tilde{H}^2_{ij}$. Furthermore, from the first summand of (3.20), we define

$$\beta_{ij} := \lim_{k\to\infty} \frac{\lambda_i^k}{\lambda_i^k + |\lambda_j^k|} \in (0,1) \text{ for } i,j \in J.$$

Since the Thibault derivative considers any appropriate $X^k \to X$, we can choose w.l.o.g. any three pairwise disjunct index sets $\tilde{I}_{(+)}, \tilde{I}_{(-)}$, and $\tilde{J}$ that fulfill $\tilde{I}_{(+)} \cup \tilde{I}_{(-)} \cup \tilde{J} = \{p+1, \ldots, p+z\}$, and a $P_M \in \mathcal{O}(X)$. Note, that $\tilde{\zeta}_{ij} = 0$ for $i, j \in \tilde{J}$ and for certain $\alpha_{ij} \in [0, +\infty[$ we get

$$\begin{aligned} M =& (\alpha_{ij})_{JJ} \circ (\tilde{H}^2)_{JJ} \\ &+ \begin{pmatrix} \tilde{H}_{\tilde{I}_{(+)}\tilde{I}_{(+)}} & \tilde{H}_{\tilde{I}_{(+)}\tilde{J}} & (\beta_{ij})_{\tilde{I}_{(+)}\tilde{I}_{(-)}} \circ \tilde{H}_{\tilde{I}_{(+)}\tilde{I}_{(-)}} \\ \tilde{H}_{\tilde{J}\tilde{I}_{(+)}} & (\star\star) & 0 \\ ((\beta_{ij})_{\tilde{I}_{(+)}\tilde{I}_{(-)}} \circ \tilde{H}_{\tilde{I}_{(+)}\tilde{I}_{(-)}})^T & 0 & 0 \end{pmatrix} \end{aligned}$$

and

$$\begin{aligned} (\star\star) = \frac{1}{2}\left(\left[\tilde{H}^2_{\tilde{J}\tilde{J}} - \left(\tilde{H}_{\tilde{J}\bar{I}} + (\tilde{H}^2)_{\tilde{J}\bar{I}}A\right)^2 + \tilde{H}^2_{\tilde{J}\bar{I}}\right]^{\frac{1}{2}} + \tilde{H}_{\tilde{J}\tilde{J}}\right) \\ -(\alpha_{ij})_{\tilde{J}\tilde{J}} \circ (\tilde{H}^2)_{\tilde{J}\tilde{J}}, \end{aligned}$$

for $\bar{I} := N \setminus \tilde{J}$. Note, that $\alpha_{ij} = \alpha_{ji}$ and the sets $\tilde{I}_{(+)}, \tilde{I}_{(-)}, \tilde{J}$, the value of each β_{ij} and α_{ij} depend on the sign and on the convergence speed of the eigenvalues in Λ^k. This completes the proof.

□

Remark 3.26. Theorem 3.25 gives us a superset of the Thibault limit set $T\Pi_+$ at X for H.

Remark 3.27. The contingent derivative lies in the Thibault limit set for Π_+. This is easily seen: If X^k is chosen such that no eigenvalue in D^k converges towards zero and $\lim_{k\to\infty} \frac{t_k}{|\gamma_i^k|} = 0$ for all $i \in I^k$, then we have $A = 0$ in $(\star\star)$ and

$$M = \lim_{k\to\infty} \frac{1}{t_k} \tilde{\Delta}(k)_{J^k J^k} = (\tilde{H}_{JJ})_+.$$

Remark 3.28. Note, that A must be bounded for any H, since from $(\star\star)$ we get

$$\tilde{H}^2_{\tilde{J}\tilde{J}} + \tilde{H}^2_{\tilde{J}\bar{I}} \succeq \left(\tilde{H}_{\tilde{J}\bar{I}} + (\tilde{H}^2)_{\tilde{J}\bar{I}}A\right)^2. \tag{3.21}$$

Remark 3.29. From the following considerations, we get a condition for the construction of the Thibault limit set. Choose $\alpha_{ij} = 0$ for $i, j \in \tilde{J}$ and $\alpha_{ij} = \lim_{k\to\infty} \frac{t_k}{|\lambda_i^k|+|\lambda_j^k|}$ for $i, j \in \tilde{I}_{(+)} \cup \tilde{I}_{(-)}$. Furthermore, let $a_{ii} = \lim_{k\to\infty} \frac{t_k}{\lambda_i^k}$

for $i \in \tilde{I}_{(+)} \cup \tilde{I}_{(-)}$ and else (for $i \neq j$) $a_{ij} = 0$ and consider (3.21). If we now construct matrices M, for all $t_k \searrow 0$, then we get a subset $\tilde{\mathcal{M}}_H \subset \mathcal{M}_H$. By replacing $\mathcal{M}_H$ with $\tilde{\mathcal{M}}_H$ in Theorem 3.25, we get a condition for the construction of $T\Pi_+(X)(H)$.

Remark 3.30. Known results of the Thibault derivative for vectors can also be deduced from Theorem 3.25. Here, we show that

$$\operatorname{Diag}(T\pi_+(x)(h)) \subset T\Pi_+(\operatorname{Diag} x)(\operatorname{Diag} h),$$

where $\pi_+(x) := x_+$ is the Euclidean projection of a vector x to the nonnegative orthant in $\mathbb{R}^n$. The following result is calculated by Klatte and Kummer [26]:

$$T\pi_+(x)(h) = \left\{ (r_1 h_1, \dots, r_n h_n) \mid \begin{array}{ll} r_i = 1 & \text{if } x_i > 0, \\ r_i = 0 & \text{if } x_i < 0, \\ r_i \in [0,1] & \text{if } x_i = 0. \end{array} \right\}.$$

Since X and H are diagonal we can write $X := \operatorname{Diag} x$ and $H := \operatorname{Diag} h$. Note, that x_i are the eigenvalues of X. For an appropriate permutation matrix P_M we get

$$T\Pi_+(X)(H) = P_M^T \begin{pmatrix} (\tilde{h}_i)_{I_p I_p} & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & 0 \end{pmatrix} P_M.$$

Clearly the matrix M is also a diagonal matrix. Let us choose the index sets as following

$$\begin{aligned} \tilde{J} &= \emptyset, \\ \tilde{I}_{(+)} &= \{ i \mid x_i = 0 \text{ and } \tilde{h}_i < 0 \}, \\ \tilde{I}_{(-)} &= \{ i \mid x_i = 0 \text{ and } \tilde{h}_i \geq 0 \}. \end{aligned}$$

For appropriate $t_k \searrow 0$, we choose $x_i^k \to x_i$ such that $\alpha_i := \lim_{k \to \infty} \frac{t_k}{2|x_i^k|}$ fulfills

$$\alpha_i \tilde{h}_i + 1 \in [0,1] \text{ if } i \in \tilde{I}_{(+)} \text{ and } \alpha_i \tilde{h}_i \in [0,1] \text{ if } i \in \tilde{I}_{(-)}.$$

Then, for

$$
\begin{aligned}
r_i &= 1 && \text{if } i \in I_{(+)}, \\
r_i &= \alpha_i \tilde{h}_i + 1 && \text{if } i \in \tilde{I}_{(+)}, \\
r_i &= \alpha_i \tilde{h}_i && \text{if } i \in \tilde{I}_{(-)}, \\
r_i &= 0 && \text{if } i \in I_{(-)}
\end{aligned}
$$

we get $\mathrm{Diag}(T\pi_+(x)(h))$.

3.3.3 Strong Regularity of the Kojima System, Simple

We are interested in the stable behavior of critical points of the SDP Kojima function $\mathcal{F}$, as defined above. In the following section, we show that the Thibault limit set provides us with sufficient and necessary conditions for strong regularity of $\mathcal{F}$.

Chainrule and Simple

According to Theorem 3.19, in order to show strong regularity of the SDP Kojima function $\mathcal{F}$ at (x^0, Y^0), we need to prove injectivity of $T\mathcal{F}(x^0, Y^0)$. As shown above, $\mathcal{F}$ can be written as the product of a Lipschitz function $M(x)$ and the matrix of projections $N(Y)$. However, in general, the product rule of differentiation for Thibault derivatives does not hold. Kummer [32] shows the following lemma for functions in $\mathbb{R}^n$.

Lemma 3.31. *Let $f(\cdot) = g(h(\cdot))$, where g and h map $\mathbb{R}^m$ into $\mathbb{R}^p$ and $\mathbb{R}^n$ into $\mathbb{R}^m$, respectively. Then, the following statements hold:*

(i) $Tf(x)(u) \subset Tg(h(x))(Th(x)(u))$.

(ii) If g is a C^1-function, then (i) holds as an equation: $Tf(x)(u) = Dg(h(x))(Th(x)(u))$.

A similar result can be achieved for a mapping between symmetric matrix spaces.
We are interested in the equality of (i) in Lemma 3.31 without assuming differentiability of g. In order to show equality for the product rule of the Thibault derivative of the SDP Kojima function, the property *simple* is helpful.
The term *simple* was first defined by Kummer [32].

Definition 3.32. A locally Lipschitz function $F\colon \mathcal{S}^p \to \mathcal{S}^q$, $p, q \in \mathbb{N}$, is called *simple* at X^0 if for all $H \in \mathcal{S}^p$ and $V \in TF(X^0)(H)$ and each sequence $s_k \searrow 0$, there is a sequence $Z^k \to X^0$ such that

$$V = \lim s_k^{-1}\left[F(Z^k + s_k H) - F(Z^k)\right]$$

holds for at least some subsequence of $k \to \infty$.

Now, we prove that Π_+ is simple for certain X^0.

Lemma 3.33. *Π_+ is simple at any X^0, if X^0 has at most one zero eigenvalue.*

Proof. Given any $H \in \mathcal{S}^n$ and $V \in T\Pi_+(X^0)(H)$, let $t_k \searrow 0$ and $X^k = (P^k)^T \Lambda^k P^k \to X^0$ be a sequence such that

$$V = \lim_{k\to\infty} t_k^{-1}\left[F(X^k + t_k H) - F(X^k)\right].$$

For any $s_k \searrow 0$ and by using the representation (3.20), we construct an appropriate sequence $Z^k \to X^0$ to attain the same V.
We take a closer look at representation (3.20) and replace t_k and X^k with s_k and Z^k. The terms in (3.20) that are of relevance are the limits of $A^k := t_k(D^k)^{-1}$, $\frac{t_k}{|\lambda_i^k|+|\lambda_j^k|}$, and $\frac{\lambda_i^k}{\lambda_i^k+|\lambda_j^k|}$.

First, assume that X^0 is nonsingular and let $Z^k := X^k$. Then,

$$\lim_k A^k = \lim_k s_k(D^k)^{-1} = 0 = \lim_k t_k(D^k)^{-1},$$

and

$$\lim_k \frac{s_k}{|\lambda_i^k| + |\lambda_j^k|} = 0 = \lim_k \frac{t_k}{|\lambda_i^k| + |\lambda_j^k|}.$$

We plug these results into (3.20), and are done.
Now, let us assume that X^0 has one zero eigenvalue $\lambda_z = 0$, i.e. $\lambda_j \neq 0$ for $j \neq z$.
If we have $\lambda_z^k = 0$ for all k, or a subsequence thereof, then we can choose $X^k = Z^k$, or a subsequence of Z^k, and get the same situation as above ($\lambda_z^k = 0$ does not appear in A^k).
Now, let us assume $\lambda_z^k \neq 0$ and construct Z^k as follows:

$$Z^k = (P^k)^T \Theta^k P^k,$$

where P^k comes from the spectral decomposition of X^k and Θ^k is a diagonal matrix, which is defined below. Then, $\tilde{H}^k$ stays the same in (3.20).
Furthermore, we know from part 2 of the proof of Theorem 3.25 that for all l satisfying $\lambda_l^k \neq 0$ (for sufficiently large k)

$$\frac{t_k}{\lambda_l^k} \to c_l \in \mathbb{R}.$$

We distinguish between $c_z \neq 0$ and $c_z = 0$, and get the following two cases:
Case 1. Suppose $c_z \neq 0$ then $\frac{s_k}{t_k}\lambda_z^k \to 0$. For $k \to \infty$, we define Θ^k with the diagonal entries $\theta_l^k, l = 1, \ldots, n$, such that

(i) $\theta_l^k = \lambda_l^k$, if $l \neq z$,

(ii) $\theta_z^k = \lambda_z^k$, if $\lambda_z^k = 0$,

(iii) $\theta_z^k = \frac{s_k}{t_k}\lambda_z^k$, if $\lambda_z^k \neq 0$.

We construct a subsequence $\theta_z^{k_z}$ of θ_z^k such that $\theta_z^{k_z}$ fulfills (iii) for all k_z. W.l.o.g. let $k := k_z$.
By construction of Z^k, it follows directly that $Z^k \to X^0$.
For the construction of $A = \lim_k A^k$, we have for $j \neq z$

$$\lim_k s_k(\theta_j^k)^{-1} = 0 = \lim_k t_k(\lambda_j^k)^{-1} \tag{3.22}$$

and for $j = z$

$$s_k(\theta_z^k)^{-1} = s_k(\frac{s_k}{t_k}\lambda_z^k)^{-1} = t_k(\lambda_z^k)^{-1}.$$

For the construction of $\frac{t_k}{|\lambda_i^k|+|\lambda_j^k|}$, we have

$$\lim_k \frac{s_k}{|\theta_i^k| + |\theta_j^k|} = 0 = \lim_k \frac{t_k}{|\lambda_i^k| + |\lambda_j^k|}. \tag{3.23}$$

Finally, for $\frac{\lambda_i^k}{\lambda_i^k+|\lambda_j^k|}$ we get (for $\lambda_i, \lambda_j \neq 0$ the case is trivial) for $i = z$

$$\lim_k \frac{\theta_z^k}{\theta_z^k + |\theta_j^k|} = \lim_k \frac{\frac{s_k}{t_k}\lambda_z^k}{\frac{s_k}{t_k}\lambda_z^k + |\lambda_j^k|} = 0 = \lim_k \frac{\lambda_z^k}{\lambda_z^k + |\lambda_j^k|}$$

and for $j = z$

$$\lim_k \frac{\theta_i^k}{\theta_i^k + |\theta_z^k|} = \lim_k \frac{\lambda_i^k}{\lambda_i^k + \frac{s_k}{t_k}|\lambda_z^k|} = 1 = \lim_k \frac{\lambda_i^k}{\lambda_i^k + |\lambda_z^k|}.$$

Case 2. Let us assume that $c_z = 0$, i.e. $\lim_k \frac{t_k}{\lambda_z^k} = 0$. Similar to case 1, we construct a subsequence of Θ^k, however, for (iii) we write $\theta_z^k = \sqrt{s_k}$. Then, $Z^k \to X^0$ is true. For the construction of A, we have (3.22) and

$$\lim_k s_k(\theta_z^k)^{-1} = \lim_k s_k(\sqrt{s_k})^{-1} = 0 = \lim_k t_k(\lambda_z^k)^{-1}.$$

For the construction of $\frac{t_k}{|\lambda_i^k|+|\lambda_j^k|}$, we have (3.23).
Finally, for $\frac{\lambda_i^k}{\lambda_i^k+|\lambda_j^k|}$ we get (for $\lambda_i, \lambda_j \neq 0$ the case is trivial) for $i = z$

$$\lim_k \frac{\theta_z^k}{\theta_z^k + |\theta_j^k|} = \lim_k \frac{\sqrt{s_k}}{\sqrt{s_k} + |\lambda_j^k|} = 0 = \lim_k \frac{\lambda_z^k}{\lambda_z^k + |\lambda_j^k|}$$

and for $j = z$

$$\lim_k \frac{\theta_i^k}{\theta_i^k + |\theta_z^k|} = \lim_k \frac{\lambda_i^k}{\lambda_i^k + \sqrt{s_k}} = 1 = \lim_k \frac{\lambda_i^k}{\lambda_i^k + |\lambda_z^k|}.$$

We plug the results of case 1 and 2 into (3.20), and are done.
Since the choice of s_k is arbitrary, the condition for simple is fulfilled, and Π_+ is simple at any X^0 with at most one zero eigenvalue. $\square$

Remark 3.34. Case 2 in the proof above is not applicable for an X^0 with two or more zero eigenvalues. Whence, another approach is necessary, which remains an open question.

Remark 3.35. The mapping $N(Y)$ is simple when $\Pi_+(Y)$ is simple. This follows from its construction, and we have

$$TN(Y)(H) = \begin{pmatrix} 0 & 0 \\ \mathrm{Diag}(\mathrm{vec}\, T\Pi_+(Y)(H), \dots) & 0 \\ 0 & 0 \\ 0 & H - T\Pi_+(Y)(H) \end{pmatrix}.$$

Simple gives us the chain rule for the Thibault derivative. For more details see Klatte and Kummer [26], and Fusek [17]. Theorem 3.36 by Klatte and Kummer [26, Corollary 6.10] gives us the equality of the product rule of the Thibault derivative. Hence, when $N(Y)$ is simple, we can calculate the Thibault derivative of the SDP Kojima function more easily.

Theorem 3.36. *(Klatte, Kummer 2002, Thibault product rule) Let $F(x,y) = M(x)N(y)$, where $M(\cdot)$ and $N(\cdot)$ are locally Lipschitz matrix-valued functions of related size. Suppose, that one of them is simple. Then the product rule of differentiation holds for TF, i.e.,*

$$TF(x,y)(u,v) = [TM(x)(u)]N(y) + M(x)[TN(y)(v)].$$

For the SDP Kojima system, M and N are locally Lipschitz. Assume N is simple. Let $(u,H) \in \mathbb{R}^n \times \mathcal{S}^p$ be a nontrivial direction, and we get

$$T\mathcal{F}(x,Y)(u,H) = [TM(x)(u)]N(Y) + M(x)[TN(Y)(H)].$$

This leads to Theorem 3.37.

Theorem 3.37. *Let $f,g \in C^{1,1}$ and N be simple at Y. The Kojima function $\tilde{\mathcal{F}}$, as given in (3.3), is strongly regular at (x,Y) if and only if*

$$0 \notin T\mathcal{F}(x,Y)(u,H) = [TM(x)(u)]N(Y) + M(x)[TN(Y)(H)] \tag{3.24}$$

for nontrivial directions (u,H). The statement in (3.24) is true if and only if for each solution of the system

$$\begin{aligned}&DG(x)(u) \in H - T\Pi_+(Y)(H),\\ &0 \in D(\operatorname{vec} G(x)^T)\operatorname{vec}(T\Pi_+(Y)(H)) + T(Df(x) + D(\operatorname{vec} G(x)^T)\operatorname{vec} Y_+)u\end{aligned}$$

we have $(u,H) = (0,0)$.
If $f,g \in C^2$, then we can write $DM(x)u$ instead of $TM(x)(u)$.

Proof. Suppose $\tilde{\mathcal{F}}$ is strongly regular at (x,Y). From Corollary 3.23, we know that regularity of $\tilde{\mathcal{F}}$ is equivalent to injectivity of $\mathcal{F}$ at (x,Y). Since $N(Y)$ is simple, Theorem 3.36 enables the application of the product rule, whence a sufficient and necessary condition for strong regularity of the Kojima system. The rest of the proof follows from the similarity to the proof of Theorem 3.14. □

Theorem 3.37 shows us a similar procedure to show first-order optimality conditions in $\mathcal{S}^n$, as already exists in $\mathbb{R}^n$ for i.a. polyhedral cones. An interesting question that remains is whether the Clarke generalized Jacobian and the Thibault limit set are the same. We do not yet have an answer to this question, but give a short overview on the topic in the next section.

3.4 Related Work

In the previous section, we saw that the injectivity of $T\mathcal{F}$ is a sufficient and necessary condition for strong regularity. Another generalized derivative, the Clarke generalized Jacobian $\partial_C\mathcal{F}$, also guarantees strong regularity if $\partial_C\mathcal{F}$ is injective at a zero. We use the following definition for the Clarke generalized Jacobian and the B-differential.

Definition 3.38. Let F be a mapping between two normed spaces. Then, $\partial_C F(x)$ is the *Clarke generalized Jacobian* if

$$\partial_C F(x) = \operatorname{conv}\{\partial_B F(x)\},$$

where $\partial_B F(x)$ is the *B-differential* of F if

$$\partial_B F(x) = \big\{ \lim_{k\to\infty} DF(x^k) \mid F \text{ is F-differential at } x^k \big\}.$$

In general, however, the Clarke generalized Jacobian is not a necessary condition for strong regularity. Clearly, if $T\mathcal{F}$ and $\partial_C\mathcal{F}$ coincide, then looking at $\partial_C\mathcal{F}$ is both sufficient and necessary for strong regularity. The Clarke generalized Jacobian can be constructed as the convex hull of $T\mathcal{F}$. Hence, if the Thibault limit set of the projection function is convex, then it coincides with $\partial_C\Pi_+$. Unfortunately, $T\Pi_+$ is a subset of the set of matrices given in Theorem 3.25.
Another approach would be to compare $T\Pi_+$ to existing calculations of the Clarke generalized Jacobian of the projection onto $\mathcal{S}^p_+$. In e.g. Malick and Sendov [36], and Chan and Sun [8] $\partial_C\Pi_+$ has been studied and even calculated.

Malick and Sendov

Malick and Sendov [36] explicitly compute the Clarke generalized Jacobian for the projection onto the cone of positive semidefinite matrices. They give a complete description with help of the tensor notation. To show their result, we need a few definitions. Similar to the construction of a diagonal matrix $\operatorname{Diag} v$ from a vector v, they have the following definition for a diagonal 4-tensor of an $n \times n$ matrix M

$$\left(\operatorname{Diag}^{(12)} M\right)_{j_1 j_2}^{i_1 i_2} = \begin{cases} M^{i_1 i_2} & \text{if } i_1 = j_2, i_2 = j_1 \\ 0 & \text{otherwise.} \end{cases}$$

Let $O(n)^X$ be the set of orthogonal matrices that give the *ordered* spectral decomposition of X, and the matrix Λ has the entries $\Lambda_{ij} := \frac{\lambda_i}{\lambda_i + |\lambda_j|}$.
The set $\mathcal{D}_{\{01\}}(z)$ is a set of $z \times z$ symmetric matrices with zeros and ones, where the entries form nonincreasing sequences (from top to bottom and left to right) as illustrated below for $\mathcal{D}_{\{01\}}(2)$

$$\mathcal{D}_{\{01\}}(2) = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right\}.$$

Theorem 3.39 (Malick, Sendov 2006)**.** *The Clarke generalized Jacobian of Π_+, the projection map onto the cone of positive semidefinite matrices, at $X \in \mathcal{S}^n$ is*

$$\partial_C \Pi_+(X) = \operatorname{conv} \left\{ P \left(Diag^{(12)} \mathcal{D}_{\{01\}}(X) \right) P^T \mid P \in O(n)^X \right\},$$

where

$$\mathcal{D}_{\{01\}}(X) := \left\{ \begin{pmatrix} 1_{p\times p} & 1_{p\times z} & \Lambda \\ 1_{z\times p} & D & 0_{z\times m} \\ \Lambda^T & 0_{m\times z} & 0_{m\times m} \end{pmatrix} \mid D \in \mathcal{D}_{\{01\}}(z) \right\},$$

and $1_{p\times p}$ is the $p \times p$ matrix with all entries equal one. p, z, and m are the number of positive, negative, and zero, respectively, eigenvalues of X.

For the Clarke generalized Jacobian $\partial_C \Pi_+(X)H$, we see a similarity with the Thibault derivative $T\Pi_+(X)(H)$. However, $\mathcal{D}_{\{01\}}(z)$ only contains zeros and ones, hence, even though the convex hull of the Thibault derivative might coincide with the Clarke generalized Jacobian for a given direction, the generating system $P(\operatorname{Diag}^{(12)} \mathcal{D}_{\{01\}}(X))P^T$ is not the Thibault limit set. An interesting approach, that remains an open problem, would be to calculate $T\mathcal{F}$ with 4-tensors, which might also enable a more precise comparison to $\partial_C \Pi_+$.

B-differential and Clarke's generalized derivative by Chan and Sun

Chan and D. Sun [8] stay in the two-dimensional space to define $\partial_C \Pi_+$, by always looking at the Clarke generalized Jacobian in a direction H. They give the same structure as we give in Theorem 3.25 for the Thibault derivative (we use the same definitions for the index sets), however, their instructions are for calculating the B-differential $\partial_B \Pi_+^{|J|}(0)$ for a $|J| \times |J|$ matrix, where $|J| := \#J$. The convex hull of $\partial_B \Pi_+^{|J|}$ is the Clarke generalized Jacobian. For an element $V \in \partial_C \Pi_+(X)$, they write

$$V(H) = P \begin{pmatrix} \tilde{H}_{I_p I_p} & \tilde{H}_{I_p J} & \Lambda \circ \tilde{H}_{I_p I_m} \\ \tilde{H}_{I_p J}^T & V_{|J|}(\tilde{H}_{JJ}) & 0 \\ \tilde{H}_{I_p J}^T \circ \Lambda^T & 0 & 0 \end{pmatrix} P^T \quad \forall H \in \mathcal{S}^n$$

and concentrate on $V_{|J|}(\tilde{H}_{JJ})$, when X is the zero matrix.
For $p(t) \equiv \max\{0, t\}$, $t \in \mathbb{R}$, they define

$$[p^{[1]}(z)]_{ij} = \begin{cases} \frac{p(z_i) - p(z_j)}{z_i - z_j} & \in [0,1] \quad z_i \neq z_j \\ p'(z_i) & \in \{0,1\} \quad z_i = z_j \end{cases}$$

and

$$\mathcal{U}_{|J|} := \left\{ \Omega \mid \Omega = \lim_{k \to \infty} p^{[1]}(z^k), \quad z^k \to 0, \quad z^k \in \mathcal{R}_>^{|J|} \right\},$$

where $\mathcal{R}_>^{|J|} := \left\{ z \in \mathbb{R}^{|J|} \mid z_1 \geq \cdots \geq z_{|J|}, \quad z_i \neq 0 \quad \forall i \right\}$.

They prove the following lemma

Lemma 3.40 (Chan, Sun 2008). *The matrix $V_{|J|}$ is an element of $\partial_B \Pi_+^{|J|}(0)$ if and only if there exists an orthogonal matrix Q and an $\Omega \in \mathcal{U}_{|J|}$ such that*

$$V_{|J|}(U) = Q \left[\Omega \circ (Q^T U Q) \right] Q^T \quad \forall U \in S^{|J|}.$$

Aside from mentioning, that $\Omega_{ij} \in [0,1]$, they do not go into further details. The Clarke generalized Jacobian follows by constructing the convex hull of $\partial_B \Pi_+^{|J|}(0)$.
In $\mathcal{U}_{|J|}$, the sequence $z^k \to 0$ is required. The values in the vectors z^k can be compared to the eigenvalues λ_i^k that converge to zero in Theorem 3.25. However, since $z_i^k \neq 0$, we do not have a comparison to the term containing $(\star\star)$

in M, and we cannot compare the results with the Thibault derivative. Looking at the convex hull of $\partial_B \Pi_+^{|J|}(0)$ gives us the Clarke generalized Jacobian, however, also here we can not compare.
Whether the Clarke generalized Jacobian in a given direction H coincides with the Thibault derivative remains an open question.

4 Calmness

Given a set valued mapping $S\colon P \rightrightarrows X$, where X is a complete metric space and P a Banach space, we are interested in *calmness* of S in a closed set $C \subset X$, a kind of Lipschitz property. In particular, we characterize calmness by means of the convergence behavior of solution methods.
We recall that S is called *calm* at a point $(\bar{p}, \bar{x}) \in \operatorname{graph} S$ if there exists $\varepsilon, \delta, L > 0$ such that for all $p \in B(\bar{p}, \delta)$ we have

$$x \in S(p) \cap B(\bar{x}, \varepsilon) \Rightarrow B(x, L\,\|p - \bar{p}\|) \cap S(\bar{p}) \neq \emptyset,$$

where $B(\bar{p}, \delta)$ is the closed δ-ball and $\operatorname{graph} S := \{(p, x) \in P \times X \mid x \in S(p)\}$.
Our basic parametrized model is

$$S(p) := \{x \in X \mid p \in F(x)\},$$

where F is a closed multifunction from X to P. S can be the solution set of e.g. an optimization problem or its constraint system.
Considering the additional constraint, that our solutions must be in a closed set C, we write

$$S_C(p) := S(p) \cap C.$$

Let S be the solution set of nonlinear inequality systems, defined by (differentiable) Lipschitz functions, and C a (convex) closed cone (however not necessarily a polyhedron), e.g. the cone of positive semidefinite matrices. In this chapter, we are, in particular, interested in calmness of solution sets S_C of such cone programs.
A considerable amount of research regarding calmness of multifunctions or metrical subregularity of the inverse (which is equivalent to calmness) has been done, dating back to the 1970s in papers by e.g. Clarke [10], Robinson [43], and Ioffe [23], or in more recent papers by e.g. Rockafellar and Wets [45], and Dontchev and Rockafellar [12]. We use the calmness property of a multifunction as introduced by Rockafellar and Wets [45]. Overviews on calmness can be found inter alia by Ioffe, Outrata, and Henrion (cf. [20, 24]). A historical overview is given in Section 4.4.

So far, we are not aware of any calculations for calmness concerning cone programs. While the approach by Ioffe and Outrata [24] concerns calmness over a set in a Banach space, the more general approach to calmness by Klatte and Kummer [28], which interests us, only defines calmness on the whole space. As Kummer [34], we consider the more general Hoelder-calmness, when possible.

This chapter is built up as follows: In Section 4.1, we look at the definition of calmness of S restricted to C, and generalize the definition of calmness, by looking at Hoelder-calmness. In Theorem 4.3, we show equivalent conditions for Hoelder-calmness, and, furthermore, demonstrate that neither S_C inherits the calmness properties of S, nor vice versa.
In Section 4.2, we look at the approach of projecting $S(p)$ onto C, and how calmness is preserved.
Finally, in our main section, Section 4.3, we interpret Theorem 4.3 and Corollary 4.8, and look at necessary and sufficient conditions for calmness of S_C. We compare stability of a generalized equation $0 \in F(x)$ to the application of solution methods, and look at examples of procedures to show calmness for semidefinite programs. This chapter is concluded with an overview on related work in Section 4.4.

4.1 Calmness

Consider a closed multifunction $S\colon P \rightrightarrows X$, where X is a complete metric space, P is a linear normed space over $\mathbb{R}$, and S is said to be closed if $\operatorname{graph} S$ is a closed set. Given a closed subset C of X, we consider $S_C(p) = S(p) \cap C$. This map plays a role below in the study of solution sets of parametric inequality systems intersected with a fixed set. Furthermore, $B(x)$ is the closed unit ball around x of the related space and $B(x, \varepsilon)$ the ε-ball.

4.1.1 General Characteristics of Calmness

The following definition generalizes calmness, which is a local Lipschitz type property of S. For $q = 1$, we get the definition of (proper) calmness.

Definition 4.1 (Hoelder-calmness)**.** Let $(\bar{p}, \bar{x}) \in \operatorname{graph} S$ and $q > 0$. S is called *calm* $[q]$ at $(\bar{p}, \bar{x})$ if $\exists \varepsilon, \delta, L > 0$:

$$x \in S(p) \cap B(\bar{x}, \varepsilon) \Rightarrow B(x, L \|p - \bar{p}\|^q) \cap S(\bar{p}) \neq \emptyset \qquad \forall p \in B(\bar{p}, \delta).$$

Note, that if $S(p) \cap B(\bar{x}, \varepsilon) = \emptyset$ for all $p \in B(\bar{p}, \delta) \backslash \{\bar{p}\}$ then S is (automatically) calm at $(\bar{p}, \bar{x})$. Of course, we can reduce the variables in Definition 4.1 by substituting δ with ε. We have $\mathrm{d}(x_a, x_b)$ for the metric distance between two points and $\mathrm{dist}_A(x_a)$ for the shortest distance of x_a to a set A.
Consider the following algorithm:

Algorithm (ALG1). *Put* $(p^1, x^1) = (p, x) \in \operatorname{graph} S$ *and choose for* $k = 1, 2, \ldots$ $(p^{k+1}, x^{k+1}) \in \operatorname{graph} S$ *in such a way that for a given* $\lambda \in (0, 1)$

$$\begin{aligned} \lambda\, \mathrm{d}(x^{k+1}, x^k) &\leq \left\|p^k - \bar{p}\right\|^q - \left\|p^{k+1} - \bar{p}\right\|^q && (4.1) \\ \left\|p^{k+1} - \bar{p}\right\| &\leq (1 - \lambda) \left\|p^k - \bar{p}\right\|. && (4.2) \end{aligned}$$

We call ALG1 *applicable* (*with respect to* S) if (p^{k+1}, x^{k+1}) exist in each step. Then, we have the following remark and results:

Remark 4.2. If ALG1 is applicable, hence for a $\lambda > 0$ and any $(p^1, x^1) \in \operatorname{graph} S$ near $(\bar{p}, \bar{x})$ there exists a p^2 and an $x^2 \in S(p^2)$ such that $\mathrm{d}(x^2, x^1) \leq \frac{1}{\lambda}(\left\|p^1 - \bar{p}\right\|^q - \left\|p^2 - \bar{p}\right\|^q)$, then by complete induction we get

$$\mathrm{d}(x^{n+1}, x^1) \leq \sum_{k=1}^{n} \mathrm{d}(x^{k+1}, x^k) \leq \frac{1}{\lambda}\left(\left\|p^1 - \bar{p}\right\|^q - \left\|p^{n+1} - \bar{p}\right\|^q\right).$$

Since $\sum_{k=1}^{n} \mathrm{d}(x^{k+1}, x^k)$ is uniformly bounded by $\left\|p^1 - \bar{p}\right\|^q / \lambda$ for each n, a Cauchy sequence $\{x^k\}_{k \geq 1}$ in X is generated. Thus, by completeness of X, there is a limit $x_{\bar{p}} := \lim_{k \to \infty} x^k \in X$. Inequality (4.2) lets the sequence $\{p^k\}_{k \geq 1}$ converge towards $\bar{p}$, and since S is closed, we get $x_{\bar{p}} \in S(\bar{p})$.

Theorem 4.3. *Let* $f \colon X \to P$ *be locally Lipschitz near* $\bar{x} \in X$, $S = f^{-1}$ *and* $q \in (0, 1]$.
Then, the following are equivalent:

(i) S *is calm* $[q]$ *at* $(\bar{p}, \bar{x}) = (f(\bar{x}), \bar{x})$.

(ii) ALG1 is applicable for some fixed $\lambda \in (0, 1)$ *and any* (p^1, x^1) *near* $(\bar{p}, \bar{x})$.

Furthermore, if S *is calm* $[q]$ *at* $(\bar{p}, \bar{x}) = (f(\bar{x}), \bar{x})$, *then the following two equivalent statements are true:*

(iii) *There exists a $\mu \in (0,1)$ such that for each x^1 near $\bar{x}$ there exists a sequence $\{x^k\}_{k\geq 1}$ with $x^k \to x_{\bar{p}} \in S(\bar{p})$ satisfying*

$$\begin{aligned} \left\|f(x^k) - f(\bar{x})\right\|^q - \left\|f(x^{k+1}) - f(\bar{x})\right\|^q \geq & \quad \mu \, \mathrm{d}(x^k, x^{k+1}) \\ \textit{and } \mathrm{d}(x^{k+1}, x^k) \geq & \quad \mu \left\|f(x^k) - f(\bar{x})\right\|. \end{aligned} \tag{4.3}$$

(iii)' *There exists a $\mu \in (0,1)$ such that for each x near $\bar{x}$ there is some x' satisfying*

$$\begin{aligned} \|f(x) - f(\bar{x})\|^q - \|f(x') - f(\bar{x})\|^q \geq & \quad \mu \, \mathrm{d}(x, x') \\ \textit{and } \mathrm{d}(x', x) \geq & \quad \mu \, \|f(x) - f(\bar{x})\|. \end{aligned} \tag{4.4}$$

For $q = 1$, all statements above are equivalent.

Proof. *(i)* $\Rightarrow$ *(ii)* If calmness $[q]$ is satisfied with related constants $L, \varepsilon, \delta > 0$ then for $p^k \in B(\bar{p}, \delta)$ we get

$$\forall x \in S(p^k) \cap B\big(\bar{x}, \varepsilon\big) \Rightarrow B(x, L \left\|p^k - \bar{p}\right\|^q) \cap S(\bar{p}) \neq \emptyset.$$

Choose $p^{k+1} = \bar{p}$; then we get

$$\left\|p^{k+1} - \bar{p}\right\| = 0 \leq (1 - \lambda) \left\|p^k - \bar{p}\right\| < \delta$$

for each $\lambda \in (0,1)$. Further, for any x^k in the local solution set $S(p^k) \cap B(\bar{x}, \varepsilon)$ there exists an x^{k+1} in $S(\bar{p})$ such that $\mathrm{d}(x^{k+1}, x^k) \leq L \left\|p^k - \bar{p}\right\|^q$, hence,

$$\lambda \, \mathrm{d}(x^{k+1}, x^k) \leq \left\|p^k - \bar{p}\right\|^q - \left\|p^{k+1} - \bar{p}\right\|^q$$

with $\lambda := \frac{1}{L}$ and ALG1 is applicable.
(ii) $\Rightarrow$ *(i)* If calmness $[q]$ is violated and $L, \varepsilon > 0$, such that $L = \frac{1}{\lambda}$, then for all $\delta > 0$ there exists a $p \in B(\bar{p}, \delta)$ and an $x \in S(p) \cap B(\bar{x}, \varepsilon)$ such that for all $x_{\bar{p}} \in S(\bar{p})$ we get $\mathrm{d}(x, x_{\bar{p}}) > L \|p - \bar{p}\|^q$. According to Remark 4.2, we have $\mathrm{d}(x, x_{\bar{p}}) \leq L \|p - \bar{p}\|^q$. Hence, (p, x) is near $(\bar{p}, \bar{x})$ but ALG1 is not applicable.
(ii) $\Rightarrow$ *(iii)* According to Remark 4.2 and $S = f^{-1}$, we can write p as $f(x)$. The first inequality in (4.3) follows from (4.1). The operator f is locally Lipschitz. Hence, the triangle inequality and (4.2) yield

$$\begin{aligned} \mathrm{d}(x^{k+1}, x^k) &\geq K \left(\left\|f(x^k) - f(\bar{x})\right\| - \left\|f(x^{k+1}) - f(\bar{x})\right\|\right) \\ &\geq \lambda \, K \left\|f(x^k) - f(\bar{x})\right\|, \end{aligned}$$

for some positive constant K and x^1 close enough to $\bar{x}$. Let $\mu := \min\{\lambda K, \lambda\}$ and the assertion follows.
(iii) $\Rightarrow$ *(iii)'* This direction follows by choosing $x := x^k$ and $x' := x^{k+1}$.
(iii)' $\Rightarrow$ *(iii)* By defining $x := x^k$ and denoting x' as x^{k+1}, one gets a sequence $\{x^k\}_{k\geq 1}$ such that (4.3) holds. For all initial points x^1 sufficiently close to $\bar{x}$,

$$\mathrm{d}(x^n, x^1) \leq \sum_{k=1}^{n-1} \mathrm{d}(x^{k+1}, x^k) \leq \frac{1}{\mu}\left(\left\|f(x^1) - f(\bar{x})\right\|^q - \|f(x^n) - f(\bar{x})\|^q\right)$$

shows that x^n belongs to an arbitrarily small neighborhood of $\bar{x}$, and $x^k \to x_{\bar{p}} \in S(\bar{p})$ follows from the argument in Remark 4.2.
If $q = 1$, then we have:
(iii) $\Rightarrow$ *(ii)* Suppose (4.3) is satisfied for the conditions given above. Let $f(x^k) := p^k$ and $\lambda = \mu^2$ then inequality (4.1) follows directly from the first inequality in (4.3). Furthermore, (4.3) yields

$$\left\|p^k - \bar{p}\right\| - \left\|p^{k+1} - \bar{p}\right\| \geq \mu^2 \left\|p^k - \bar{p}\right\|,$$

which is equivalent to

$$\left\|p^{k+1} - \bar{p}\right\| \leq (1 - \lambda) \left\|p^k - \bar{p}\right\|.$$

□

Remark 4.4. For $q < 1$, we generally do not have *(iii)* $\Rightarrow$ *(ii)* in Theorem 4.3. From (4.3) we only get (4.2) and $\left\|p^{k+1} - \bar{p}\right\| \leq (1 - \lambda_k) \left\|p^k - \bar{p}\right\|$ for a $\lambda_k \to 0$ when $k \to \infty$.

Remark 4.5. Taking into account, that the proof of Corollary 2 in [28] also works for complete metric spaces X – not only for Banach spaces as shown by Klatte and Kummer [28] – Theorem 4.3 coincides with Corollary 2 [28] in the case $q = 1$. Further, note that Kummer's Proposition 3.3 in [34] applied to $S = f^{-1}$ is very similar to our Theorem 4.3, however, we are interested in a statement which generalizes Corollary 2 [28] and therefore added our proof of Theorem 4.3 for completeness.

Remark 4.6. If $q = 1$ and we have calmness as given in Theorem 4.3, then $f(x^k)$ converges linearly towards $f(\bar{x})$, since

$$\left\|f(x^{k+1}) - f(\bar{x})\right\| \leq (1 - \mu^2) \left\|f(x^k) - f(\bar{x})\right\|.$$

Remark 4.7. For a locally Lipschitz operator f, $S = f^{-1}$ is not calm $[q]$ for $q > 1$.

Theorem 4.3 directly applies to the multifunction S_C, since a closed subset C of a complete metric space X is itself a complete metric subspace of X with the induced metric, and we immediately have Corollary 4.8.

Corollary 4.8. *Let C be a closed subset of X and $f\colon X \to P$ be locally Lipschitz on C near $\bar{x} \in C$, and let $S = f^{-1}$ and $q \in (0,1]$. Then, the following statements are equivalent:*

(i) S_C is calm $[q]$ at $(\bar{p}, \bar{x}) = (f(\bar{x}), \bar{x})$.

(ii) ALG1 is applicable with respect to S_C for some fixed $\lambda \in (0,1)$ and any (p^1, x^1) near $(\bar{p}, \bar{x})$.

Furthermore, if S_C is calm $[q]$ at $(\bar{p}, \bar{x}) = (f(\bar{x}), \bar{x})$, then the following two equivalent statements are true:

(iii) There exists a $\mu \in (0,1)$ such that for each $x^1 \in C$ near $\bar{x}$ there exists a sequence $\{x^k\}_{k\geq 1} \subset C$ with $x^k \to x_{\bar{p}} \in S_C(\bar{p})$ satisfying (4.3).

(iii)' There exists a $\mu \in (0,1)$ such that for each $x \in C$ near $\bar{x}$ there is some $x' \in C$ satisfying (4.4).

For $q = 1$, all statements above are equivalent.

Remark 4.9. If $\bar{x} \in \operatorname{int} C$, the interior of C, then calmness of S is equivalent to calmness of S_C (easily proven by choosing an $\varepsilon > 0$ sufficiently small).

4.1.2 Calmness of S versus S_C

In this section, we show that in a linear space calmness of S and S_C do not have to be related. Calmness of S_C does not necessarily imply calmness of S, as the following Example 4.10 shows. Nor is the reverse the case, as can be seen in Example 4.11.

Example 4.10. Given is the multifunction $S\colon \mathbb{R}_+ \rightrightarrows \mathbb{R}_+$ with $S(p) = \{\sqrt{p}\}$ and $C = \{0\}$. Then, S_C is calm at $(0,0)$ since for $\varepsilon, \delta > 0$ $S_C(p) \cap B(0,\varepsilon) = \emptyset$ for all $p \in (0,\delta]$, but S is not calm at $(0,0)$.

In Example 4.11, we consider the space of symmetric matrices (however, shorter examples in $\mathbb{R}^2$ such as Example 4.13 would also fulfill the purpose), since our initial motivation is to look at sensitivity analysis for semidefinite programming problems.

Example 4.11. In the space of real symmetric matrices $\mathcal{S}^2$, the following system is given:

$$\left\langle \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} x_{11} & x_{12} \\ x_{12} & x_{22} \end{pmatrix} \right\rangle = t,$$

$$\left\langle \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} x_{11} & x_{12} \\ x_{12} & x_{22} \end{pmatrix} \right\rangle \equiv 1.$$

This system is valid for all matrices with the specific entries $x_{11} = t, x_{22} = 1$. Let C be the set of all positive semidefinite symmetric matrices $\mathcal{S}^2_+$ and the norm be the maximum absolute row sum norm $\|.\|_\infty$.
Assuming $X_0 \in C$, we look at calmness for S and for S_C at $(0, X_0)$.
Calm for S: The solution set $S(t)$ is the set of all symmetric matrices that fulfill the system from above:

$$S(t) := \left\{ \begin{pmatrix} t & x_{12} \\ x_{12} & 1 \end{pmatrix} \middle| \, x_{12} \in \mathbb{R} \right\}.$$

It is easily shown, that this solution set is calm at $(0, X_0)$ for any $X_0 \in S(0)$: Take any $\tilde{x}_{12} \in \mathbb{R}$ and denote

$$\tilde{X} = \begin{pmatrix} t & \tilde{x}_{12} \\ \tilde{x}_{12} & 1 \end{pmatrix}.$$

For $\tilde{X}_0 \in S(0)$ with

$$\tilde{X}_0 = \begin{pmatrix} 0 & \tilde{x}_{12} \\ \tilde{x}_{12} & 1 \end{pmatrix},$$

we get $\left\| \tilde{X} - \tilde{X}_0 \right\|_\infty = |t|$. Hence, S is calm at $(0, X_0)$.
Calm for S_C: The next question is whether S_C is calm at $(0, X_0)$, and C is the set of positive semidefinite matrices. We look at the solution set in C which is, for $t \geq 0$,

$$S_C(t) := \left\{ \begin{pmatrix} t & x_{12} \\ x_{12} & 1 \end{pmatrix} \middle| \, |x_{12}| \leq \sqrt{t} \right\},$$

and consequently the solution set $S_C(0)$ is a singleton containing

$$X_0 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then, for $t > 0$ and

$$\tilde{X} = \begin{pmatrix} t & \sqrt{t} \\ \sqrt{t} & 1 \end{pmatrix}$$

one has for any $L > 0$ and t sufficiently small

$$\left\|\tilde{X} - X_0\right\|_\infty = \left\|\begin{pmatrix} t & \sqrt{t} \\ \sqrt{t} & 0 \end{pmatrix}\right\|_\infty = t + \sqrt{t} > L\,t.$$

Hence, S_C is not calm at $(0, X_0)$.

Remark 4.12. In the above Example 4.11, S_C is calm $[q]$ at $(0, X_0)$ for $q = \frac{1}{2}$.

Example 4.13. Consider $S(p) := \{(x, y) \in \mathbb{R}^2 \mid y \leq p\}$ and

$$C := \{(x, y) \in \mathbb{R}^2 \mid y \geq x^2\}$$

and the Euclidean norm. Then, S is calm at $(0, 0)$, since for any $(x, y) \in S(p) \cap B((0, 0), \varepsilon)$ and $y > 0$ ($y \leq 0$ is trivial) we get

$$\operatorname{dist}((x, y), S(0)) = y \leq p = Lp$$

for $L = 1$. However, in the case of $S_C(p)$ where $S_C(0) = \{(0, 0)\}$, we can choose for any $L > 0$ a $p > 0$ such that

$$(\sqrt{p}, p) \in S_C(p) \cap B((0, 0), \varepsilon)$$

but $\operatorname{dist}((\sqrt{p}, p), S_C(0)) > Lp$. Hence, S_C is not calm at $(0, 0)$.

4.2 Projection of S onto C

Here, our focus is to study calmness of mappings that project the sets $S(p)$ onto C, the set of solutions in C closest to $S(p)$. In this section, let $S\colon P \rightrightarrows X$ be a closed multifunction, where X is a real Hilbert space with the metric $\mathrm{d}(\cdot, \cdot)$ induced by the inner product and P a linear normed space over $\mathbb{R}$, with some appropriate norm $\|\cdot\|$, respectively. Let C be a closed convex set in X. Furthermore, we have the projection mapping

$$\pi_C(x) := \operatorname{argmin}\{\mathrm{d}(x, c) \mid c \in C\}$$

and the multifunction

$$\operatorname{proj}_C(A) := \{\pi_C(a) \mid a \in A\}$$

for any $A \subset X$.
As an example, consider a mapping $S = F$ with $F(\bar{p}) \cap C = \emptyset$ and $F(p) = \{(x, y) \in \mathbb{R}^2 \mid f(x) - y \leq p\}$, where f is a continuous function on $\mathbb{R}$ and $C \subset \mathbb{R}^2$ is a closed convex set. Note, that if F is calm $[q]$ at $(\bar{p}, \bar{x})$ then so is $\operatorname{proj}_C F$ at $(\bar{p}, \pi)$ for a $\pi \in \pi_C(\bar{x})$. We show that this sufficient condition is in fact not true in general, however, a slightly weaker condition is.
In this section, we state and prove a projection theorem with two corollaries and a helping lemma. The lemma shows us the inheritance of calmness $[q]$ to a smaller neighborhood. In the theorem, we prove that the calmness properties of a mapping at a point outside of C are inherited when projecting both the image of the mapping and the point onto C, however, the projection is restricted to a local part of the image and certain given conditions. With the help of examples, we show that neither a generalization nor the inverse of the theorem is possible.
Consider $S_\varepsilon(p) := S(p) \cap B(\bar{x}, \varepsilon)$.

Lemma 4.14. *If S is calm $[q]$ at $(\bar{p}, \bar{x})$ then so is S_ε for sufficiently small $\varepsilon > 0$.*

Proof. If S is calm $[q]$ at $(\bar{p}, \bar{x})$ then $\exists \varepsilon, \delta, L > 0$ such that $\forall p \in B(\bar{p}, \delta)$ we get

$$x \in S_\varepsilon(p) \quad \Rightarrow \quad S(\bar{p}) \cap B(x, L\|p - \bar{p}\|^q) \neq \emptyset.$$

We choose $\delta \leq (\frac{\varepsilon}{2L})^{\frac{1}{q}}$ and apply this δ from here on.
For any $x \in S_{\frac{\varepsilon}{2}}(p)$, we get $S(\bar{p}) \cap B(x, L\|p - \bar{p}\|^q) \neq \emptyset$ for all $p \in B(\bar{p}, \delta)$. Thus, there is an $\tilde{x} \in S(\bar{p})$ such that

$$\mathrm{d}(\tilde{x}, x) \leq L\|p - \bar{p}\|^q \leq L\delta^q \leq \tfrac{\varepsilon}{2}.$$

Since $\mathrm{d}(x, \bar{x}) \leq \frac{\varepsilon}{2}$, we have $\mathrm{d}(\tilde{x}, \bar{x}) \leq \varepsilon$ and $S_\varepsilon(\bar{p}) \cap B(x, L\|p - \bar{p}\|^q) \neq \emptyset$ for all $p \in B(\bar{p}, \delta)$. Hence, S_ε is calm $[q]$ at $(\bar{p}, \bar{x})$. □

Remark 4.15. In Lemma 4.14, both X and P can be considered as metric spaces. However, for the following proofs it is required that X be a Hilbert space.

Theorem 4.16. *If S is calm $[q]$ at $(\bar{p}, \bar{x})$ then for $\operatorname{proj}_C S_\varepsilon$ at $(\bar{p}, \pi_C(\bar{x}))$, with $\varepsilon > 0$ sufficiently small, there exists $\delta, L > 0$ such that*

$$\begin{aligned} &\pi \in \operatorname{proj}_C S_{\frac{\varepsilon}{2}}(p) \Rightarrow \\ &\quad \operatorname{proj}_C S_\varepsilon(\bar{p}) \cap B(\pi, L\|p - \bar{p}\|^q) \neq \emptyset \quad \forall p \in B(\bar{p}, \delta). \end{aligned} \tag{4.5}$$

Proof. Since S is calm $[q]$ at $(\bar{p},\bar{x})$, it follows from Lemma 4.14 that for sufficiently small $\varepsilon > 0$, S_ε is calm $[q]$ at $(\bar{p},\bar{x})$, too. This means that there are $\delta, L > 0$, such that for all $p \in B(\bar{p},\delta)$ we have

$$x \in S_\varepsilon(p) \cap B\big(\bar{x}, \frac{\varepsilon}{2}\big) \Rightarrow S_\varepsilon(\bar{p}) \cap B\big(x, L\,\|p-\bar{p}\|^q\,\big) \neq \emptyset.$$

Since π_C is a projection onto a convex set, we have

$$\mathrm{d}\big(\pi_C(x_a), \pi_C(x_b)\big) \leq \mathrm{d}(x_a, x_b) \quad \forall x_a, x_b \in X.$$

Then, for any $\pi \in \operatorname{proj}_C S_{\frac{\varepsilon}{2}}(p)$ there is an $x \in S_{\frac{\varepsilon}{2}}(p)$ with $\pi := \pi_C(x)$, and we get

$$\operatorname{dist}_{\operatorname{proj}_C S_\varepsilon(\bar{p})}\big(\pi\big) \leq \operatorname{dist}_{S_\varepsilon(\bar{p})}(x) \leq L\,\|p-\bar{p}\|^q \quad \forall p \in B(\bar{p},\delta).$$

This completes the proof. □

It is necessary to look at S_ε and not at the projection of all elements of S, since S might contain points which are not near $\bar{x}$, but have projections close to $\pi_C(\bar{x})$ as in Example 4.17.

Example 4.17. The multifunction $S\colon \mathbb{R} \rightrightarrows \mathbb{R}^2$ is defined as follows

$$\begin{aligned}
S(\bar{p}) &= \big\{(0,0),(0,b)\big\}, \\
S(p) &= \big\{[-\sqrt{p},\sqrt{p}] \times \{p+b\}\big\} \quad \text{for } p > 0, \\
S(p) &= \emptyset \quad \text{for } p < 0,
\end{aligned}$$

for a positive $b > 0$ and $\bar{p} = 0$. For $\bar{x} = (0,0)$, we easily see that S is calm at $(\bar{p},\bar{x})$, hence, it fulfills property (4.5), since $S(p) \cap B(\bar{x},\varepsilon)$ is empty for $p \neq \bar{p}$ and ε sufficiently small. If we look at the projection onto $C = \mathbb{R} \times \{0\}$, then the projection of $S_{\tilde{\varepsilon}}$ for a sufficiently small $\tilde{\varepsilon}$ is calm at $\big(\bar{p},\pi_C(\bar{x})\big)$.
However, for sufficiently small ε and p ($\sqrt{p} < \varepsilon$) we can choose an

$$x \in \operatorname{proj}_C S(p) \cap B\big(\pi_C(\bar{x}),\varepsilon\big) = \big\{[-\sqrt{p},\sqrt{p}] \times \{0\}\big\}$$

with $x = \sqrt{p}$ such that $\operatorname{dist}_{\operatorname{proj}_C S(\bar{p})}(x) = \sqrt{p} > Lp$ for any $L > 0$, whence $\operatorname{proj}_C S$ is not calm at $\big(\bar{p},\pi_C(\bar{x})\big)$.

In the following Corollary 4.18, a sufficient condition for calmness is given, such that property (4.5) can be replaced by calmness in Theorem 4.16.

Corollary 4.18. *Assume S is calm $[q]$ at $(\bar{p}, \bar{x})$. Furthermore let $\varepsilon^*, \delta > 0$ be sufficiently small, such that for any $\varepsilon' \in (0, \varepsilon^*]$ there exists a $\mu' > 0$ such that*

$$B(\pi_C(\bar{x}), \mu') \cap (\operatorname{proj}_C B(\bar{x}, \varepsilon') \setminus \operatorname{proj}_C B(\bar{x}, \frac{\varepsilon'}{2})) = \emptyset, \tag{4.6}$$

and for all $p \in B(\bar{p}, \delta)$

$$\pi \in \operatorname{proj}_C B(\bar{x}, \frac{\varepsilon'}{2}) \cap \operatorname{proj}_C S(p) \Rightarrow \exists x \in S(p) \cap B(\bar{x}, \frac{\varepsilon'}{2}) : \pi_C(x) = \pi. \tag{4.7}$$

Then $\operatorname{proj}_C S_\varepsilon$ is calm $[q]$ at $(\bar{p}, \pi_C(\bar{x}))$ for sufficiently small $\varepsilon > 0$.

Proof. We choose ε' and μ' sufficiently small. If $\pi \in \operatorname{proj}_C(S(p) \cap B(\bar{x}, \varepsilon')) \cap B(\pi_C(\bar{x}), \mu')$ then from (4.6) we get $\pi \in \operatorname{proj}_C B(\bar{x}, \frac{\varepsilon'}{2})$. Since we clearly have $\pi \in \operatorname{proj}_C S(p)$, we obtain from (4.7) that there exists an $x \in B(\bar{x}, \frac{\varepsilon'}{2}) \cap S(p)$ such that $\pi_C(x) = \pi$. Following the proof of Theorem 4.16, with $\varepsilon := \varepsilon'$, we get, for any $\pi \in \operatorname{proj}_C S_\varepsilon(p) \cap B(\pi_C(\bar{x}), \mu')$ there is an $x \in S_\varepsilon(p) \cap B(\bar{x}, \frac{\varepsilon}{2})$ with $\pi = \pi_C(x)$, and we get for an $L > 0$

$$\operatorname{dist}_{\operatorname{proj}_C S_\varepsilon(\bar{p})}(\pi) \leq \operatorname{dist}_{S_\varepsilon(\bar{p})}(x) \leq L \|p - \bar{p}\|^q \quad \forall p \in B(\bar{p}, \delta).$$

This completes the proof. □

A special case of Corollary 4.18 is when C is the cone of positive semidefinite symmetric matrices $C = \mathcal{S}^n_+$.

Corollary 4.19. *Let $S \colon X \to \mathcal{S}^n$, $C := \mathcal{S}^n_+$ and S be calm $[q]$ at $(\bar{Y}, \bar{X})$. Furthermore, let $\delta > 0$ be sufficiently small, such that for any $\varepsilon' > 0$ (4.7) is fulfilled. Then $\operatorname{proj}_C S_\varepsilon$ is calm $[q]$ at $(\bar{Y}, \bar{X}_+)$ for a sufficiently small $\varepsilon > 0$.*

We recall, that in Lemma 2.4, we showed that for matrices $A, B \in \mathcal{S}^n$ we have

$$A = (A + B)_+ \iff B = (A + B)_- \iff A \succeq 0, B \preceq 0, \langle A, B \rangle = 0. \tag{4.8}$$

Proof. It suffices to show that for sufficiently small $\varepsilon > 0$ there exists $\mu > 0$ such that

$$A_{\mu,\varepsilon} := B\left(\bar{X}_+, \mu\right) \cap (\operatorname{proj}_C B(\bar{X}, \varepsilon) \setminus \operatorname{proj}_C B(\bar{X}, \frac{\varepsilon}{2})) = \emptyset.$$

For $\bar{X} \succeq 0$ (i.e. $\bar{X} \in C = \mathcal{S}^n_+$) and $\mu := \frac{\varepsilon}{2}$, we get

$$B(\bar{X}, \mu) \cap \operatorname{proj}_C B(\bar{X}, \varepsilon) \subseteq B(\bar{X}, \mu) \cap C \subseteq \operatorname{proj}_C B(\bar{X}, \frac{\varepsilon}{2})$$

and $A_{\mu,\varepsilon} = \emptyset$.
For $\bar{X} \prec 0$ choose ε sufficiently small, then $\text{proj}_C B(\bar{X}, \varepsilon) = 0$, and it follows directly that $A_{\mu,\varepsilon} = \emptyset$.
For $\bar{X} \not\succeq 0$ and $\bar{X} \not\prec 0$ assume that $A_{\mu,\varepsilon} \neq \emptyset$ for all $\varepsilon > 0$ and $\mu > 0$, then there exist $Z^i \in A_{\mu_i,\varepsilon}$ such that

$$Z^i \to \bar{X}_+$$

when $\mu_i \to 0$, but clearly $Z^i \neq \bar{X}_+$ for all i.
Furthermore, there exist $\bar{Z}^i \in B(\bar{X}, \varepsilon) \setminus B^\circ(\bar{X}, \frac{\varepsilon}{2})$ with the same orthogonal matrices in their spectral decomposition as for $\bar{X}$, such that $\bar{Z}^i_+ = Z^i$, and w.l.o.g.

$$\bar{Z}^i \to X^* \in B(\bar{X}, \varepsilon) \setminus B^\circ(\bar{X}, \frac{\varepsilon}{2}),$$

where $B^\circ(\bar{X}, \frac{\varepsilon}{2})$ is the open $\frac{\varepsilon}{2}$-ball, because of compactness. For the projection of this limit, we get $X^*_+ = \bar{X}_+$, since the projection is a continuous mapping. Now, let $\bar{X} = P\Lambda P^T$ be a spectral decomposition of $\bar{X}$, P an orthogonal matrix and Λ the diagonal matrix with the eigenvalues $\lambda_i(\bar{X})$ sorted in descending order. We choose an appropriate $s = (s_1, \ldots, s_n)$ where $s_i \in (-\varepsilon, \varepsilon)$, $\sum_i s_i^2 \in [\frac{\varepsilon}{2}, \varepsilon]$ and $s_i = 0$ if $\lambda_i(\bar{X}) \geq 0$, and write

$$X^* = P(\Lambda + \text{Diag}(s))P^T \in B(\bar{X}, \varepsilon) \setminus B^\circ(\bar{X}, \frac{\varepsilon}{2})$$

(if necessary choose a smaller ε), which is possible because $\bar{Z}^i \to X^* \in B(\bar{X}, \varepsilon)$ and $\bar{Z}^i_+ = Z^i \to \bar{X}$. W.l.o.g. we have a subsequence $X^k \to X^*$, $X^k \in B(\bar{X}, \varepsilon) \setminus B(\bar{X}, \frac{\varepsilon}{2})$ and $X^k_+ \neq \bar{X}_+$ with

$$X^k := \beta^k X^* + (1 - \beta^k)\bar{X} \quad \text{for } \beta^k > 1, \quad \lim_{k\to\infty} \beta^k = 1$$

and we get $X^k = P\Lambda^k P^T$.
Note, that $X^* = \bar{X}_+ + X^*_-$ and $\langle \bar{X}_+, X^*_- \rangle = 0$. Then, we get

$$\begin{aligned} X^k =& \beta^k(\bar{X}_+ + X^*_-) + (1 - \beta^k)(\bar{X}_+ + \bar{X}_-) \\ =& \bar{X}_+ + \beta^k X^*_- + (1 - \beta^k)\bar{X}_-. \end{aligned} \tag{4.9}$$

It follows directly that

$$\langle \bar{X}_+, \beta^k X^*_- + (1 - \beta^k)\bar{X}_- \rangle = \beta^k \langle \bar{X}_+, X^*_- \rangle + (1 - \beta^k)\langle \bar{X}_+, \bar{X}_- \rangle = 0.$$

Furthermore, since X^k and $\bar{X}$ share the same orthogonal matrices, and for sufficiently small $\varepsilon > 0$, we have $\lambda_i(\bar{X}) + s_i < 0$ if $\lambda_i < 0$ and we get

$$\beta^k X_-^* + (1 - \beta^k)\bar{X}_- \preceq 0$$

for sufficiently large k since $X_-^* \neq 0$. From (4.8) and (4.9), the contradiction $X_+^k = \bar{X}_+$ follows. Hence, $A_{\mu,\varepsilon} = \emptyset$ for sufficiently small μ and ε. This concludes the proof. $\square$

Remark 4.20. Corollary 4.19 also holds for $\mathbb{R}_+^n$.

If $\operatorname{proj}_C S_\varepsilon$ is calm at $\big(\bar{p}, \pi(\bar{x})\big)$, then that does not necessarily mean that S is calm at a point $(\bar{p}, \bar{x})$. This is easily shown by considering C as the origin.

4.3 Calmness Conditions for Inequality Systems

For a finite set of locally Lipschitzian functions $g_i \colon X \to \mathbb{R}$, where X is a complete metric space and $i \in I = \{1, \dots, m\}$, and a closed set $C \subseteq X$ we study calmness $[q]$ of the following multifunction $\Sigma_C \colon \mathbb{R}^m \rightrightarrows X$

$$\Sigma_C(p) := \{x \in C \mid g_i(x) \leq p_i \quad \forall i \in I\}$$

at $(\bar{p}, \bar{x})$ with a finite index set I.
One possibility is to include $\operatorname{dist}_C(x)$ to the inequality system describing Σ_C, and define

$$\tilde{\Sigma}(\tilde{p}) := \{x \in X \mid (g(x), \operatorname{dist}_C(x)) \leq \tilde{p}\},$$

where $\tilde{p} \in \mathbb{R}^{m+1}$. Calmness of Σ_C is even equivalent to calmness of $\tilde{\Sigma}$, as shown in Lemma 4.21. However, the mapping $\operatorname{dist}_C(x)$ is not differentiable, and in the Banach space we focus on continuously differentiable g_i to look at a necessary and sufficient algorithmic approach for calmness (in the classic sense, where $q = 1$) and the speed of convergence in such a case.

Lemma 4.21. *Let X, C, Σ_C, and g be given as mentioned above and $q \in (0, 1]$. Let $\bar{x} \in \Sigma_C(0)$. Then, at $(0, \bar{x})$ the mapping Σ_C is calm $[q]$ if and only if at $(0, \bar{x})$ the mapping $\tilde{\Sigma}$ is calm $[q]$.*

The following proof of Lemma 4.21 was given by an anonymous reviewer.

Proof. The if-part being evident, let Σ_C be calm at $(0, \bar{x})$. Then, for some $L, \varepsilon > 0$,

$$\mathrm{dist}_{\Sigma_C(0)}(x) \le L\,\|p\|^q \quad \forall p \in B(0,\varepsilon) \quad \forall x \in \Sigma_C(p) \cap B(\bar{x},\varepsilon). \tag{4.10}$$

Let $K > 0$ be a Lipschitz constant of g on $B(\bar{x},\delta)$ for some $\delta > 0$. Put $\tilde{\varepsilon} := \min\{\frac{\varepsilon}{4}, \frac{\varepsilon}{4K}, \frac{\delta}{4}, 1\} > 0$.
For arbitrary $\tilde{p} \in B(0,\tilde{\varepsilon})$ in $\mathbb{R}^{m+1}$ and $x \in \tilde{\Sigma}(\tilde{p}) \cap B(\bar{x},\tilde{\varepsilon})$, we write $\tilde{p} = (p,t)$ and consider the maximum norm in $\mathbb{R}^{m+1}$, hence, $x \in B(\bar{x}, \frac{\delta}{2})$, $g(x) \le p$, and $\mathrm{dist}_C(x) \le t \le \tilde{\varepsilon} \le \frac{\delta}{4}$.
Choose $\tilde{x} \in C$ such that $\mathrm{d}(x,\tilde{x}) \le 2t$. Then, $\mathrm{d}(x,\tilde{x}) \le \frac{\delta}{2}$ and $\tilde{x} \in B(\bar{x},\delta)$, whence $\|g(x) - g(\tilde{x})\| \le K\,\mathrm{d}(x,\tilde{x})$. Since $g(\tilde{x}) \le p + (g(\tilde{x}) - g(x))$, it follows that $\tilde{x} \in \Sigma_C(u)$ where $u := p + (g(\tilde{x}) - g(x))$. Since $\|p\| \le \frac{\varepsilon}{2}$ and $|t| \le \frac{\varepsilon}{4K}$, we get

$$\|u\| \le \|p\| + K\,\mathrm{d}(x,\tilde{x}) \le \frac{\varepsilon}{2} + K2t \le \varepsilon, \tag{4.11}$$

whence $u \in B(0,\varepsilon)$.
Finally, from $\mathrm{d}(x,\bar{x}) \le \frac{\varepsilon}{4}, |t| \le \frac{\varepsilon}{4}$ and $\mathrm{d}(x,\tilde{x}) \le 2t \le \frac{\varepsilon}{2}$, we see that $\tilde{x} \in B(\bar{x},\varepsilon)$. This allows us to apply (4.10) with respect to $\tilde{x}$. Combined with (4.11) and the fact that $t < 1$ by definition of $\tilde{\varepsilon}$, we get

$$\begin{aligned}
\mathrm{dist}_{\tilde{\Sigma}(0)}(x) &\le \mathrm{d}(x,\tilde{x}) + \mathrm{dist}_{\Sigma_C(0)}(\tilde{x}) \le 2t + L\,\|u\|^q \\
&\le 2t + L(\|p\| + K2t)^q \le 2t^q + L(\|\tilde{p}\| + 2K\,\|\tilde{p}\|)^q \\
&\le [2 + L(1+2K)^q]\,\|\tilde{p}\|^q\,.
\end{aligned}$$

This shows calmness of $\tilde{\Sigma}$ at $(0,\tilde{x})$. ∎

4.3.1 Convergence Speed for a Calm System

Let $g^m(x) := \max_{i\in I} g_i(x)$, $\alpha_+ := \max\{0,\alpha\}$ for $\alpha \in \mathbb{R}$ and the relative slack of $g_i(x)$ in comparison with $g^m(x)$ be $s_i(x) := \max\{0, \frac{g^m(x) - g_i(x)}{g^m(x)}\}$.
We define $\tilde{\Sigma}_C(r) := \{x \in C \mid (g^m)_+(x) = r\}$.

Lemma 4.22. *Σ_C is calm $[q]$ at $(0,\bar{x})$ iff $\tilde{\Sigma}_C$ is calm $[q]$ at $(0,\bar{x})$.*

Proof. W.l.o.g. we consider the maximum norm in $\mathbb{R}^m$. Obviously, $\Sigma_C(0) = \tilde{\Sigma}_C(0)$.

($\Rightarrow$) If Σ_C is calm $[q]$ at $(0,\bar{x})$ then we have positive ε, δ and L that, for all $p \in B(0,\delta)$, satisfy

$$x \in B(\bar{x},\varepsilon) \text{ and } g_i(x) \le p_i \quad \forall i \in I \quad \Rightarrow \quad \mathrm{dist}_{\Sigma_C(0)}\big(x\big) \le L\,\|p\|^q\,.$$

Then, for all $r \in [0,\delta]$ (for negative r calmness is trivial), we choose a $p \in B(0,\delta)$ with $p_i = r$ for all i and get

$$\begin{aligned} &x \in B(\bar{x},\varepsilon) \text{ and } g_i(x) \le (g^m)_+(x) = r \\ &\qquad\qquad \Rightarrow \quad \mathrm{dist}_{\Sigma_C(0)}\big(x\big) \le L\,\|p\|^q = L\,|r|^q\,, \end{aligned}$$

which proves the if-part.
($\Leftarrow$) Suppose, $\tilde{\Sigma}_C$ is calm $[q]$ but Σ_C is not calm $[q]$ at $(0,\bar{x})$. Then, for all positive ε, δ and L there exists some $\tilde{x} \in B(\bar{x},\varepsilon)$ and $p \in B(0,\delta)$ that satisfy $g_i(\tilde{x}) \le p_i$ for all i and $\mathrm{dist}_{\Sigma_C(0)}\big(\tilde{x}\big) > L\,\|p\|^q$. We define $\tilde{p}$ such that $\tilde{p}_i = \min\{p_i, (g^m)_+(\tilde{x})\}$.
Then, we get $(g^m)_+(\tilde{x}) = \max\{\tilde{p}_i\} =: \tilde{r}$ but

$$\mathrm{dist}_{\Sigma_C(0)}\big(\tilde{x}\big) > L\,\|p\|^q \ge L\,|\tilde{r}|^q\,,$$

which contradicts the assumption. □

Similarly to Klatte and Kummer [28, Theorem 3], we get the following necessary respectively equivalent condition for calmness for Σ_C. In Theorem 4.24, we consider $q = 1$.

Theorem 4.23. *Let $g^m(\bar{x}) = 0$. If Σ_C is calm $[q]$ at $(0,\bar{x})$, then there exists some $\mu \in (0,1)$ and a neighborhood Ω of $\bar{x}$ such that the following holds: for any $x \in \Omega \cap C$ with $g^m(x) > 0$ sufficiently small, we have some $x' \in C$ with $x \neq x'$ such that*

$$\begin{aligned} \frac{(g_i)_+(x')^q - (g_i)_+(x)^q}{\mathrm{d}(x,x')} &\le \frac{g^m(x)^q - (g_i)_+(x)^q}{\mathrm{d}(x,x')} - \mu \quad \forall i \in I \\ \text{and} \quad \mu\, g^m(x) &\le \mathrm{d}(x,x') \le \frac{1}{\mu} g^m(x)^q. \end{aligned} \tag{4.12}$$

Proof. Σ_C is calm $[q]$ at $(0,\bar{x})$ if and only if $\tilde{\Sigma}_C$ is calm $[q]$ at $(0,\bar{x})$ as shown in Lemma 4.22. This means by Corollary 4.8 $(iii)'$ that if Σ_C is calm $[q]$ then for some $\mu \in (0,1)$ and any $x \in C$ near $\bar{x}$ that fulfills $g^m(x) > 0$ there is some $x' \in C$ such that

$$g^m(x)^q - (g^m)_+(x')^q \ge \mu\, \mathrm{d}(x,x') \quad \text{and} \quad \mathrm{d}(x,x') \ge \mu\, g^m(x). \tag{4.13}$$

We have

$$(g^m)_+(x')^q \leq g^m(x)^q - \mu \,\mathrm{d}(x,x') \iff$$
$$(g_i)_+(x')^q \leq g^m(x)^q - \mu \,\mathrm{d}(x,x') \quad \forall i \in I.$$

Hence, the inequalities (4.13) are, for sufficiently small $g^m(x)$, equivalent to

$$\begin{array}{rcl} \mu\, g^m(x) & \leq & \mathrm{d}(x,x') \leq \frac{1}{\mu} g^m(x)^q \\ \text{and} \quad \frac{(g_i)_+(x')^q - (g_i)_+(x)^q}{\mathrm{d}(x,x')} & \leq & \frac{g^m(x)^q - (g_i)_+(x)^q}{\mathrm{d}(x,x')} - \mu \quad \forall i \in I \end{array}$$

whence the assumption in (4.12).

□

Theorem 4.24. *Let $g^m(\bar{x}) = 0$. Then Σ_C is calm at $(0,\bar{x})$ if and only if there exists some $\mu \in (0,1)$ and a neighborhood Ω of $\bar{x}$ such that the following holds: for any $x \in \Omega \cap C$ with $g^m(x) > 0$, we have some $x' \in C$ with $x \neq x'$ such that*

$$\begin{aligned} \frac{(g_i)_+(x') - (g_i)_+(x)}{\mathrm{d}(x,x')} &\leq \frac{g^m(x) - (g_i)_+(x)}{\mathrm{d}(x,x')} - \mu \quad \forall i \in I \\ \textit{and} \quad \mu\, g^m(x) &\leq \mathrm{d}(x,x') \leq \frac{1}{\mu} g^m(x). \end{aligned} \tag{4.14}$$

Moreover, let us assume X is a Banach space, C a closed convex set, and $g \in C^1(X)$ with $g^m(\bar{x}) = 0$. Then, Σ_C is calm at $(0,\bar{x})$ if and only if there exists some $\mu \in (0,1)$ and a neighborhood Ω of $\bar{x}$ such that the following holds: for any $x \in \Omega \cap C$ with $g^m(x) > 0$, we have some $u \in \mathrm{bd}\, B(0)$ and $t > 0$ such that $x + tu \in C$ and

$$Dg_i(\bar{x})u \leq \frac{s_i(x)}{\mu} - \mu \quad \forall i \in I. \tag{4.15}$$

Proof. Σ_C is calm at $(0,\bar{x})$ if and only if $\tilde{\Sigma}_C$ is calm at $(0,\bar{x})$ as shown in Lemma 4.22. This means by Corollary 4.8 $(iii)'$ that Σ_C is calm if and only if for some $\mu \in (0,1)$ and any $x \in C$ near $\bar{x}$ that fulfills $g^m(x) > 0$ there is some $x' \in C$ such that

$$g^m(x) - (g^m)_+(x') \geq \mu \,\mathrm{d}(x,x') \quad \text{and} \quad \mathrm{d}(x,x') \geq \mu\, g^m(x).$$

These inequalities are equivalent to

$$\begin{aligned} \mu\, g^m(x) &\leq \mathrm{d}(x,x') \leq \frac{1}{\mu} g^m(x) \\ \text{and} \quad \frac{(g_i)_+(x') - (g_i)_+(x)}{\mathrm{d}(x,x')} &\leq \frac{g^m(x) - (g_i)_+(x)}{\mathrm{d}(x,x')} - \mu \quad \forall i \in I, \end{aligned}$$

whence the assumption in (4.14).
Let X be a Banach space with the norm induced metric $\mathrm{d}(x,x') := \|x - x'\|$. Here, we can consider $g_i(x)$ instead of $(g_i)_+(x)$ in (4.14). For all $x \in \Omega \cap C$, there exists a $u \in \operatorname{bd} B(0)$ and $t > 0$ such that $x + tu = x' \in C$ and

$$\begin{aligned} \frac{g_i(x+tu) - g_i(x)}{t} &\leq \frac{(g_i)_+(x+tu) - g_i(x)}{t} \leq \frac{g^m(x) - g_i(x)}{t} - \mu \\ &\leq \frac{g^m(x) - g_i(x)}{\mu\, g^m(x)} - \mu \\ &= \frac{s_i(x)}{\mu} - \mu \quad \forall i \in I, \quad t \in \left[\mu, \tfrac{1}{\mu}\right] g^m(x). \end{aligned}$$

For $x \to \bar{x}$ and $t \searrow 0$, we have uniform convergence and get

$$\sup_{i \in I, \|u\|=1} \left| \frac{g_i(x+tu) - g_i(x)}{t} - Dg_i(\bar{x})u \right| \to 0,$$

whence

$$Dg_i(\bar{x})u \leq \frac{s_i(x)}{\mu} - \mu \quad \forall i \in I \tag{4.16}$$

for a possibly smaller μ.
The reverse direction, that (4.14) follows from (4.15), is proven by applying $t = \mu\, g^m(x)$ to the equation (4.15), looking at the definition of $Dg_i(\bar{x})$ without the limes and choosing a possibly smaller μ. □

Remark 4.25. Consider g^m, μ, and Ω as in Theorem 4.24. Note, that if for all $x \in \Omega \cap C$ with $g^m(x) > 0$ there exists an $x' \in X$, $x' \neq x$, such that (4.14) is fulfilled but $x' \in C$ is not guaranteed for x sufficiently close to $\bar{x}$, then we have a necessary however not sufficient condition for calmness of Σ_C.

For a more general approach to calmness $[q]$ and applications to C^2 systems for $q = \frac{1}{2}$, we refer to Kummer [34], who proves the following theorem which includes Clarke's generalized Jacobian ∂_C.

Theorem 4.26 (Kummer [34] Theorem 4.11)**.** *$\Sigma_{\mathbb{R}^n}$ is calm $[\frac{1}{2}]$ at $(0,\bar{x})$ if $0 \notin \partial_C g^m(\bar{x})$.*

However, Σ_C is not necessarily calm $[\frac{1}{2}]$ at $(0,\bar{x})$ for an appropriate C even if $0 \notin \partial_C g^m$, as shown in Example 4.27.

Example 4.27. Let $g(x,y) = y + \cos x$, $(\bar{x},\bar{y}) = (\pi, 1)$ and $C = \mathbb{R} \times \{1\}$. Furthermore, we have $\Sigma_C(p) = \{(x,1) \mid 1 + \cos x \leq p\}$ and $\Sigma_C(0) = \{(x,1) \mid \cos x \leq -1\} = \{((2z+1)\pi, 1) \mid z \in \mathbb{Z}\}$.
Since $g^m(x,y) = g(x,y) = y + \cos x$, we have $g^m \in C^2$ and Clarke's generalized Jacobian coincides with the F-derivative, hence

$$\partial_C g^m(x,y) = (-\sin x, 1).$$

Clearly, $0 \notin \partial_C g^m(\pi, 1) = (0,1)$. However, for any $\delta > 0$, we can choose $p \in B(0,\delta)$ and $(x,1) \in \Sigma_C(p) \cap B((\pi,1),\varepsilon)$, $\varepsilon > 0$, such that $1 + \cos x = p$, and for $L > 0$ we get

$$\begin{aligned} \operatorname{dist}_{\Sigma_C(0)}\big((x,1)\big) &= \|(x,1) - (\pi,1)\| = |x - \pi| \\ &> L\, |\cos x - \cos \pi|^{\frac{1}{2}} = L\, |p|^{\frac{1}{2}}\,. \end{aligned}$$

So Σ_C is not calm $[\frac{1}{2}]$ at $\big(0, (\bar{x},\bar{y})\big)$.

In the remaining part of Section 4.3.1 we assume that X is a Banach space and C a closed convex set in X.

Remark 4.28. If we add $\varepsilon \|x - \bar{x}\|^2$ to all $g_i(x)$, then we get a perturbed system in C (cf. Klatte and Kummer [28]). If the perturbed system is calm at $(0,\bar{x})$ in X, then the original system is also calm in C.

Remark 4.29. Let the conditions be given as in Theorem 4.24. If $x' \in C$ and $x' + tu \in C$ then $x' + \mu\, g^m(x')u \in C$ since $\mu\, g^m(x') \leq t$ and C is convex.

Algorithm (ALG2)**.** *Given $x^k \in C \subset X$ and $\mu_k > 0$ for sufficiently large k, solve the system*

$$\begin{aligned} Dg_i(x^k)u \leq \frac{s_i(x^k)}{\mu_k} - \mu_k \quad \forall i, \quad \|u\| = 1, \\ x^k + \mu_k\, g^m(x^k)u \in C. \end{aligned}$$

Having a solution u, put	$x^{k+1} = x^k + \mu_k\, g^m(x^k)u,$	$\mu_{k+1} = \mu_k$
otherwise put	$x^{k+1} = x^k,$	$\mu_{k+1} = \frac{1}{2}\mu_k.$

Theorem 4.30. *Let $g \in C^1(X)$ and C be closed and convex. Σ_C is calm at $(0, \bar{x})$ if and only if there exists some $\alpha > 0$ such that for $\|x^1 - \bar{x}\|$ small enough and $x^1, \bar{x} \in C$, and $\mu_1 = 1$, it follows $\mu_k \geq \alpha$ for all k, for an appropriate u, such that (ALG2) is applicable.*
In this case, the sequence x^k converges to some $x_\pi \in \Sigma_C(0)$ and

$$g^m(x^{k+1}) < (1 - \beta^2)\, g^m(x^k) \text{ whenever } 0 < \beta < \alpha \text{ and } g^m(x^k) > 0.$$

Proof. We assume calmness of Σ_C at $(0, \bar{x})$, which is, according to Corollary 4.8, for $f(x) := g^m(x)$, equivalent to the inequalities

$$\mathrm{d}(x^{k+1}, x^k) \geq \alpha\; g^m(x^k) \text{ and } g^m(x^k) - g^m(x^{k+1}) \geq \alpha\; \mathrm{d}(x^{k+1}, x^k)$$

for appropriate $\{x^k\} \subset C$ and a fixed $\alpha > 0$. Hence, for every x^k we can choose a $\mu_k \in [\alpha, \mu_{k-1}]$ with $\mu_1 := 1$. With Theorem 4.24 and x^1 possibly closer to $\bar{x}$, we get the inequality of ALG2. This proves the sufficient condition and $x^k \to x_\pi \in \Sigma_C(0)$. The necessary condition follows directly from Theorem 4.24.
The estimate follows from Theorem 4.24 (4.14) which is equivalent to

$$g^m(x^{k+1}) \leq g^m(x^k) - \alpha t \text{ and } \alpha^2 g^m(x^k) \leq \alpha t \leq g^m(x^k),$$

where $g^m(x^k) > 0$.
Then, for all β with $0 < \beta < \alpha$, we have

$$g^m(x^{k+1}) \leq (1 - \alpha^2) g^m(x^k) < (1 - \beta^2) g^m(x^k),$$

whence $g^m(x^{k+1}) < (1 - \beta^2)^{k+1} g^m(x^1)$ and $g^m(x^k) \to 0$ for $k \to \infty$. □

To get a convex auxiliary system, it suffices to claim $\|u\| \leq 1$ in ALG2. We get a theorem similar to Theorem 4.30 with the new constant $D = 1 + \sup_i \|Dg_i(\bar{x})\|$ and the estimate

$$g^m(x^{k+1}) < (1 - \beta^2) g^m(x^k) \text{ whenever } 0 < \beta < \frac{\alpha^2}{D} \text{ and } g^m(x^k) > 0. \quad (4.17)$$

The proof is similar to the proof of Theorem 4 in [28].

4.3.2 Applications to Semidefinite Programming

The algorithm ALG2 solves problems that require the solution to be in a closed convex set. This makes semidefinite programming (SDP) problems an obvious client, considering the nonpolynomial cone of positive semidefinite matrices as C. In this section, we give a few examples of frameworks for procedures to solve SDP problems, provided that we have calmness.
The following SDP problem is given

$$\min f(X) \quad \text{s.t. } X \in \mathcal{S}^n_+,\ g_r(X) \le 0 \quad r = 1 \dots m. \tag{4.18}$$

The functions $f, g = (g_1, \dots, g_m)$ are specified below, $\mathcal{S}^n_+$ is again the set of real symmetric positive semidefinite matrices.
First, we concentrate on the constraint set of the problem, considering either C^1-functions or only linear functions. Finally, we look at the complete optimization problem and a procedure to calculate the Karush-Kuhn-Tucker points.

C^1 constraints

Let the constraints g_r be any C^1-functions, then the maximum function g^m and the slack s_r are defined as shown above. Suppose X^k is a sequence in $\mathcal{S}^n_+$ near a feasible point. The multifunction $\Sigma_C(p) := \{X \in \mathcal{S}^n \mid g_r(X) \le p_r \quad \forall r\}$ is calm at a solution if and only if the sequence X^k converges to a feasible point at a convergence speed that fulfills inequality (4.17). In other words, the following inequalites are true for a sufficiently small μ_k and a $U \in \mathcal{S}^n, \|U\| \le 1$:

$$\langle Dg_r(X^k), U \rangle \le \frac{s_r(X^k)}{\mu_k} - \mu_k \quad \forall r.$$

Hence, we are looking for u_{ij}, $i = 1 \dots n, j = i \dots n$ such that

$$\begin{aligned} \sum_{i=1}^{n} \frac{\partial g_r(X^k)}{\partial x_{ii}} u_{ii} + 2 \sum_{i=1}^{n} \sum_{j>i} \frac{\partial g_r(X^k)}{\partial x_{ij}} u_{ij} \le \frac{s_r(X^k)}{\mu_k} - \mu_k \quad \forall r \\ \sum_{i=1}^{n} u_{ii}^2 + 2 \sum_{i=1}^{n} \sum_{j>i} u_{ij}^2 \le 1. \end{aligned} \tag{4.19}$$

The quadratic inequality comes from

$$\begin{aligned}\sum_{i=1}^{n}\sum_{j=1}^{n} u_{ij}^2 &= tr(U^2) = tr(P\Lambda P^T P\Lambda P^T)\\ &= tr(P\Lambda^2 P^T) = \sum_{i=1}^{n}\lambda_i^2 = \|U\|^2,\end{aligned}$$

where $U = P\Lambda P^T$ is the spectral decomposition of U. P is an orthogonal matrix and Λ the diagonal matrix with the eigenvalues λ_i of U.
We get an inequality system with $\frac{(n+1)n}{2}$ variables and $m+1$ inequalities.

Linear constraints

From problem (4.18), let the constraints be linear functions. If we only look at the constraint set, then we get the following equation

$$g_r(X) = \langle A^r, X\rangle \quad A^r \in \mathcal{S}^n_+.$$

Clearly, $g^m(X) := \max_r \langle A^r, X\rangle$ denotes the maximum of all functions g_r. The choice of A^r in g^m depends on X as Example 4.31 illustrates.
In this case, a direction $U = (u_{ij})_{i,j\in N}$ where $N := \{1,\dots,n\}$, as above and $A^r = (a^r_{ij})_{i,j\in N}$ must fulfill

$$\langle A^r, U\rangle \le \frac{s_r(X^k)}{\mu_k} - \mu_k \quad \forall r.$$

Hence, we are looking for u_{ij}, $i = 1\dots n, j = i\dots n$ such that

$$\begin{aligned}\sum_{i=1}^{n} a^r_{ii}u_{ii} + 2\sum_{i=1}^{n}\sum_{j>i} a^r_{ij}u_{ij} &\le \frac{s_r(X^k)}{\mu_k} - \mu_k \quad \forall r\\ \sum_{i=1}^{n} u_{ii}^2 + 2\sum_{i=1}^{n}\sum_{j>i} u_{ij}^2 &\le 1.\end{aligned} \tag{4.20}$$

We have an inequality system with $\frac{(n+1)n}{2}$ variables and $m+1$ inequalities.
Example 4.31. We want to show that the choice of A^r in g^m depends on the choice of X. By choosing

$$A^1 = \begin{pmatrix}1 & 0\\ 0 & 2\end{pmatrix}, A^2 = \begin{pmatrix}2 & 0\\ 0 & 1\end{pmatrix}, X^1 = \begin{pmatrix}3 & 0\\ 0 & 2\end{pmatrix}, X^2 = \begin{pmatrix}2 & 0\\ 0 & 3\end{pmatrix},$$

we get $\max_r \langle A^r, X^1 \rangle = \langle A^2, X^1 \rangle$ but $\max_r \langle A^r, X^2 \rangle = \langle A^1, X^2 \rangle$.

Karush-Kuhn-Tucker points

In the following case, the entire problem (4.18) is considered. For this optimization problem, the Karush-Kuhn-Tucker (KKT) points $(X, y) \in \mathcal{S}^n_+ \times \mathbb{R}^m$ fulfill

$$\begin{aligned} & D_X L(X,y) = D_X f(X) + \sum_{r=1}^{m} y_r D_X g_r(X) \in -\mathcal{N}_{\mathcal{S}^n_+}(X), \quad X \in \mathcal{S}^n_+ \\ & g(X) \le 0, \quad y \ge 0, \quad \langle y, g_r(X) \rangle = 0, \end{aligned}$$

where $\mathcal{N}_{\mathcal{S}^n_+}(X)$ is the normal cone of $\mathcal{S}^n_+$ at X. This can be rewritten as

$$\begin{aligned} & D_X L(X,y), X \in \mathcal{S}^n_+, \\ & \quad h^1(X,y) = \langle X, D_X L(X,y) \rangle && \le 0, \\ & \quad h^2(X,y) = -\langle X, D_X L(X,y) \rangle && \le 0, \\ & \quad h^3(X,y) = g(X) && \le 0, \\ & \quad h^4(X,y) = -y && \le 0, \\ & \quad h^5(X,y) = \langle y, g_r(X) \rangle && \le 0, \\ & \quad h^6(X,y) = -\langle y, g_r(X) \rangle && \le 0. \end{aligned}$$

Let f and g be twice differentiable, $h = (h^1, \dots, h^6)$. We define

$$\Sigma_{\mathcal{S}^n_+}(p) := \{(X, y, D_X L(X,y)) \in \mathcal{S}^n_+ \times \mathbb{R}^m \times \mathcal{S}^n_+ \mid h(X,y) \le p\}.$$

We then get the following theorem:

Theorem 4.32. $\Sigma_{\mathcal{S}^n_+}$ *is calm at* $(X, y, D_X L(X,y))$ *if and only if a minimizing sequence* $(X^k, y^k, D_X L(X^k, y^k)) \in \mathcal{S}^n_+ \times \mathbb{R}^m \times \mathcal{S}^n_+$ *exists, that converges towards a KKT point (including* $D_X L$*) and for* $h^m(X^k, y^k)$*, the maximum of all functions, the convergence properties given in* (4.17) *are fulfilled. This means, there exist sufficiently small* $\mu_k > \alpha$ *for a fixed* $\alpha > 0$ *and* $(U, v) \in S^n \times \mathbb{R}^m$ *with* $\|(U, v)\| \le 1$ *such that the following inequalities for* (X^k, y^k) *are true:*

$$\begin{aligned} D_X h_i(X^k, y^k)(U, v) &\le \frac{s_i(X^k)}{\mu_k} - \mu_k \\ (X^{k+1}, y^{k+1}) := (X^k, y^k) + \mu_k h^m(X^k, y^k)(U, v) &\in \mathcal{S}^n_+ \times \mathbb{R}^m \\ D_X L((X^{k+1}, y^{k+1})) &\in \mathcal{S}^n_+. \end{aligned}$$

Now the next point of the sequence is (X^{k+1}, y^{k+1}).

4.4 Related Work

The definition *calmness* was first used by Clarke [10] in the context of optimal value functions in 1976, while Robinson [43] introduced the multifunction property *upper Lipschitzian* in 1979 which precedes *calmness at* $\bar{x}$ *for* $\bar{u}$. This is a calmness variant of the Aubin property, as introduced by Rockafellar and Wets [45], whose calmness property of a multifunction we use in the current chapter. Ioffe [23] introduced and characterized the property presently called *metric subregularity* (which is equivalent to calmness of the inverse) for single valued mappings, and Dontchev and Rockafellar [12] suggested the terminology *metric subregularity*.
Several different approaches have been made to analyze calmness for multifunctions, such as constraint set mappings, bilevel programs, and MPECs (cf. Henrion and Surowiec [21]).
Henrion and Outrata [19] prove a sufficient criteria for calmness of the mapping $S(y) = \{x \in \Omega \mid g(x) + y \in \Lambda\}$, where Ω is a neighborhood in X and $\Lambda \subset Y$ a closed subset. It is based on Morduhkovich's approach to the Aubin property (or metric regularity of the inverse) via the so-called limiting (Mordukhovich) coderivative (cf. Morduhkovich [39]), and guarantees calmness, under mild assumptions on Ω and g. However, if $\bar{x}$ is not a boundary point of the considered neighborhood Ω in X, then $\bar{x}$ must either be an isolated point or the condition has the Aubin property. Henrion and Outrata [20] give weaker conditions for calmness, but only in relation to special applications. Furthermore, an equivalence between calmness and the Abadie constraint qualification is shown when $\bar{x}$ is the only local solution.
These, however, are mainly sufficient but not necessary conditions for calmness. Ioffe and Outrata [24] show a sufficient condition for calmness of set valued finite dimensional spaces with help of a newly introduced derivative-like outer coderivative.
For mappings with closed and convex graphs, when replacing the limiting coderivative with the outer coderivative, one gets an equivalence for metric subregularity instead of metric regularity. If convexity of the graph is not given, then we still have a sufficient condition for calmness.
Zheng and Ng [54] look at generalized equations for a closed multifunction between Banach spaces. They show that calmness for the intersection of a

closed, convex multifunction S and a convex set A can be characterized with the help of the normal cone of S and A in a point near the solution, and the coderivative of S^{-1}. Furthermore, Zheng and Ng [55] look at a sufficient condition for calmness in the nonconvex case. Assuming $F : X \to Y$ convex, they get a necessary condition for metric subregularity, and by considering X an Asplund space and Y a finite Hilbert space or a reflexive Banach space, the sufficient and necessary conditions can be characterized with more general coderivatives.
Klatte and Kummer [27, 28] show the equivalence between the calmness property and the convergence behavior of a procedure for solving a constraint problem for multifunctions in arbitrary Banach spaces. Kummer [34] extended the analysis for calmness to multifunctions mapping into complete metric spaces and for the more general Hoelder-calmness.

So far, we are not aware of any work on sufficient and necessary conditions for calmness in cone programs, such as SDP problems. Ioffe and Outrata [24] observe calmness over a set in a Banach space, hence calmness for a solution set intersected with a cone could be regarded. But equivalent conditions to show calmness, as proven by Klatte and Kummer [28] only apply to calmness on solution sets of inequality systems.

5 Conclusion

This thesis is set out to explore sensitivity analysis in semidefinite programming. Our aim is to look at the stability of solution sets of disturbed SDP problems by rewriting the problems as parametrized optimization problems with a right-hand parameter. We seek to extend the toolbox for solution methods in SDP by characterizing strong regularity, upper regularity, and calmness in this field.

5.1 Results

In this thesis, two fields of mathematics are combined, namely sensitivity analysis and semidefinite programming. While sensitivity analysis for standard optimization problems with polyhedral cone constraints is an exhausted field, working with semidefinite programming problems is still unfolding.
The main results are chapter specific and summarized within the respective chapters. In this section, we illustrate the results of our research.
In the first approach, sensitivity of SDP problems is characterized with the help of generalized derivatives. This approach over Kojima functions is still quite new in SDP, and we introduce the product form $\mathcal{F}(x, Y) = M(x)N(Y)$, which is new for $Y \in \mathcal{S}^p$. We show upper and strong regularity of the SDP Kojima functions by looking at the solutions of a certain given system.
Furthermore, in the second approach, we are interested in showing a sufficient and necessary condition for calmness at feasible and stationary points of SDP problems. Results on calmness are even new for polyhedral cone programming. We extend an algorithmic structure to compare calmness with convergence conditions to SDP, by observing calmness and Hölder-calmness for the solution set of an inequality system intersected with a fixed algebraic constraint. Our results give sufficient and necessary conditions for calmness, and the framework for an algorithmic approach to show calmness at feasible and solution points of SDP problems. A few examples are given to demonstrate the results.

While working on the thesis research problems, we obtain several useful and new results. We appear to be the first to look at the contingent derivative of the Kojima system in the SDP context. Furthermore, we are neither aware of any previous attempts to calculate the Thibault derivative for the projection function Π_+ onto the cone of positive semidefinite matrices, nor looking at the Thibault derivative of the Kojima system, nor showing the *simple* characteristic thereof.
In the course of looking at calmness regarding a fixed algebraic constraint, we gain knowledge on calmness, or a weakened form thereof, for projection functions π_C onto convex sets C.

5.2 Open Questions

As a direct consequence of our study, several limitations were encountered, which encourage future research. Let us give a few questions that remain open or have arisen in the process of our work.

- Can we give a more precise description of the Thibault limit set of the projection function Π_+, which is described in Chapter 3.3? Would an approach over tensors, similar to Malick and Sendov's [36] calculation of the Clarke generalized Jacobian, be useful?
- Is the projection function Π_+ simple at a matrix with more than one zero eigenvalue?
- Are the Clarke generalized Jacobian and the Thibault limit set the same for the matrix projection function?
- Can we get more precise results for calmness of the projection function π_C from Chapter 4.2, for convex sets that are not semidefinite cones?
- In Chapter 4.3.1, we had to reduce our results to (proper) calmness in order to get a sufficient condition to show calmness in Theorem 4.24. Can we achieve similar results for Hölder-calmness?

While these questions do remain open, in this thesis we successfully expand results from polyhedral cone programming to SDP problems and contribute to closing the gap of its sensitivity analysis.

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